3.6 Forced Harmonic Motion: Resonance

In this section we study the motion of a damped harmonic oscillator that is subjected to a periodic driving force by an external agent. Suppose a force of the form $F_0 \cos \omega t$ is exerted upon such an oscillator. The equation of motion is

$$m\ddot{x} = -kx - c\dot{x} + F_0 \cos \omega t \quad (3.6.1)$$

The most striking feature of such an oscillator is the way in which it responds as a function of the driving frequency even when the driving force is of fixed amplitude. A remarkable phenomenon occurs when the driving frequency is close in value to the natural frequency $\omega_0$ of the oscillator. It is called resonance. Anyone who has ever pushed a child on a swing knows that the amplitude of oscillation can be made quite large if even the smallest push is made at just the right time. Small, periodic forces exerted on oscillators at frequencies well above or below the natural frequency are much less effective; the amplitude remains small. We initiate our discussion of forced harmonic motion with a qualitative description of the behavior that we might expect. Then we carry out a detailed analysis of the equation of motion (Equation 3.6.1), with our eyes peeled for the appearance of the phenomenon of resonance.

We already know that the undamped harmonic oscillator, subjected to any sort of disturbance that displaces it from its equilibrium position, oscillates at its natural frequency, $\omega_0 = \sqrt{k/m}$. The dissipative forces inevitably present in any real system changes the frequency of the oscillator slightly, from $\omega_0$ to $\omega_1$, and cause the free oscillation to die out. This motion is represented by a solution to the homogeneous differential equation (Equation 3.4.1, which is Equation 3.6.1 without the driving force present). A periodic
driving force does two things to the oscillator: (1) It initiates a "free" oscillation at its natural frequency, and (2) it forces the oscillator to vibrate eventually at the driving frequency \( \omega \). For a short time the actual motion is a linear superposition of oscillations at these two frequencies, but with one dying away and the other persisting. The motion that dies away is called the transient. The final surviving motion, an oscillation at the driving frequency, is called the steady-state motion. It represents a solution to the inhomogeneous equation (Equation 3.6.1). Here we focus only upon the steady-state motion, whose anticipated features we describe below. To aid in the descriptive process, we assume for the moment that the damping term \(-c\dot{x}\) is vanishingly small. Unfortunately, this approximation leads to the physical absurdity that the transient term never dies out—a rather paradoxical situation for a phenomenon described by the word transient! We just ignore this difficulty and focus totally upon the steady-state description, in hopes that the simplicity gained by this approximation gives us insight that helps when we finally solve the problem of the driven, damped oscillator.

In the absence of damping, Equation 3.6.1 can be written as

\[
mx + kx = F_0 \cos \omega t
\]

(3.6.2)

The most dramatic feature of the resulting motion of this driven, undamped oscillator is a catastrophically large response at \( \omega = \omega_0 \). This we shall soon see, but what response might we anticipate at both extremely low (\( \omega \ll \omega_0 \)) and high (\( \omega \gg \omega_0 \)) frequencies? At low frequencies, we might expect the inertial term \( mx \) to be negligible compared to the spring force \(-kx\). The spring should appear to be quite stiff, compressing and relaxing very slowly, with the oscillator moving pretty much in phase with the driving force. Thus, we might guess that

\[
x = A \cos \omega t
\]

At high frequencies the acceleration should be large, so we might guess that \( mx \) should dominate the spring force \(-kx\). The response, in this case, is controlled by the mass of the oscillator. Its displacement should be small and 180° out of phase with the driving force, because the acceleration of a harmonic oscillator is 180° out of phase with the displacement. The veracity of these preliminary considerations emerge during the process of obtaining an actual solution.

First, let us solve Equation 3.6.2, representing the driven, undamped oscillator. In keeping with our previous descriptions of harmonic motion, we try a solution of the form

\[
x(t) = A \cos(\omega t - \phi)
\]

Thus, we assume that the steady-state motion is harmonic and that in the steady state it ought to respond at the driving frequency \( \omega \). We note, though, that its response might differ in phase from that of the driving force by an amount \( \phi \). \( \phi \) is not the result of some initial condition! (It does not make any sense to talk about initial conditions for a steady-state solution.) To see if this assumed solution works, we substitute it into Equation 3.6.2, obtaining

\[-m \omega^2 A \cos(\omega t - \phi) + kA \cos(\omega t - \phi) = F_0 \cos \omega t\]
This works if \( \phi \) can take on only two values, 0 and \( \pi \). Let us see what is implied by this requirement. Solving the above equation for \( \phi = 0 \) and \( \pi \), respectively, yields
\[
A = \frac{F_0/m}{(\omega_0^2 - \omega^2)} \quad \phi = 0 \quad \omega < \omega_0
\]
\[
= \frac{F_0/m}{(\omega^2 - \omega_0^2)} \quad \phi = \pi \quad \omega > \omega_0
\]
We plot the amplitude \( A \) and phase angle \( \phi \) as functions of \( \omega \) in Figure 3.6.1. Indeed, as can be seen from the plots, as \( \omega \) passes through \( \omega_0 \), the amplitude becomes catastrophically large, and, perhaps even more surprisingly, the displacement shifts discontinuously from being in phase with the driving force to being 180° out of phase. True, these results are not physically possible. However, they are idealizations of real situations. As we shall soon see, if we throw in just a little damping, at \( \omega \) close to \( \omega_0 \) the amplitude becomes large but finite. The phase shift “smooths out”; it is no longer discontinuous, although the shift is still quite abrupt.

(Note: The behavior of the system mimics our description of the low-frequency and high-frequency limits.)

The 0° and 180° phase differences between the displacement and driving force can be simply and vividly demonstrated. Hold the lighter end of a pencil or a pair of scissors (closed) or a spoon delicately between forefinger and thumb, squeezing just hard enough that it does not drop. To demonstrate the 0° phase difference, slowly move your hand back and forth horizontally in a direction parallel to the line formed between your forefinger and thumb. The bottom of this makeshift pendulum swings back and forth in phase with the hand motion and with a larger amplitude than the hand motion. To see the 180°

![Figure 3.6.1](image-url)
phase shift, move your hand back and forth rather rapidly (high frequency). The bottom of the pendulum hardly moves at all, but what little motion it does undergo is 180° out of phase with the hand motion.

The Driven, Damped Harmonic Oscillator

We now seek the steady-state solution to Equation 3.6.1, representing the driven, damped harmonic oscillator. It is fairly straightforward to solve this equation directly, but it is algebraically simpler to use complex exponentials instead of sines and/or cosines. First, we represent the driving force as

\[ F = F_0 e^{i \omega t} \]

so that Equation 3.6.1 becomes

\[ m \ddot{x} + c \dot{x} + kx = F_0 e^{i \omega t} \]

The variable \( x \) is now complex, as is the applied force \( F \). Remember, though, that by Euler's identity the real part of \( F \) is \( F_0 \cos \omega t \). If we solve Equation 3.6.4 for \( x \), its real part will be a solution to Equation 3.6.1. In fact, when we find a solution to the above complex equation (Equation 3.6.4), we can be sure that the real parts of both sides are equal (as are the imaginary parts). It is the real parts that are equivalent to Equation 3.6.1 and, thus, the real, physical situation.

For the steady-state solution, let us, therefore, try the complex exponential

\[ x(t) = Ae^{i(\omega t - \phi)} \]

where the amplitude \( A \) and phase difference \( \phi \) are constants to be determined. If this "guess" is correct, we must have

\[ m \frac{d^2}{dt^2} Ae^{i(\omega t - \phi)} + c \frac{d}{dt} Ae^{i(\omega t - \phi)} + kAe^{i(\omega t - \phi)} = F_0 e^{i \omega t} \]

be true for all values of \( t \). Upon performing the indicated operations and canceling the common factor \( e^{i \omega t} \), we find

\[ -m \omega^2 A + i \omega c A + kA = F_0 e^{i \phi} = F_0 (\cos \phi + i \sin \phi) \]

Equating the real and imaginary parts yields the two equations

\[ A(k - m \omega^2) = F_0 \cos \phi \]
\[ c \omega A = F_0 \sin \phi \]

Upon dividing the second by the first and using the identity \( \tan \phi = \sin \phi/\cos \phi \), we obtain the following relation for the phase angle:

\[ \tan \phi = \frac{c \omega}{k - m \omega^2} \]

\(^{5}\)For a proof of Euler's identity, see Appendix D.
By squaring both sides of Equations 3.6.7a and adding and employing the identity 
\[ \sin^2 \phi + \cos^2 \phi = 1, \]
we find
\[ A^2 (k - m \omega^2)^2 + c^2 \omega^2 A^2 = F_0^2 \]  
(3.6.7c)

We can then solve for \( A \), the amplitude of the steady-state oscillation, as a function of the 
driving frequency:
\[ A(\omega) = \frac{F_0}{\sqrt{[(k - m \omega^2)^2 + c^2 \omega^2]^2}} \]  
(3.6.7d)

In terms of our previous abbreviations \( \omega_0^2 = k/m \) and \( \gamma = c/2m \), we can write the expres-
sions in another form, as follows:
\[ \tan \phi = \frac{2\gamma \omega}{\omega_0^2 - \omega^2} \]  
(3.6.8)
\[ A(\omega) = \frac{F_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2 \omega^2}} \]  
(3.6.9)

A plot of the above amplitude \( A \) and phase difference \( \phi \) versus driving frequency 
\( \omega \) (Fig. 3.6.2) reveals a fetching similarity to the plots of Figure 3.6.1 for the case of 
the undamped oscillator. As can be seen from the plots, as the damping term 
approaches 0, the resonant peak gets larger and narrower, and the phase shift sharpens up, 
ultimately approaching infinity and discontinuity, respectively, at \( \omega_0 \). What is 
not so obvious from these plots is that the amplitude resonant frequency is not \( \omega_0 \) when 
damping is present (although the phase shift always passes through \( \pi/2 \) at \( \omega_0 \)).
Amplitude resonance occurs at some other value \( \omega_\ast \), which can be calculated by dif-
ferentiating \( A(\omega) \) and setting the result equal to zero. Upon solving the resultant equa-
tion for \( \omega \), we obtain
\[ \omega_\ast^2 = \omega_0^2 - 2\gamma^2 \]  
(3.6.10)

\( \omega_\ast \) approaches \( \omega_0 \) as \( \gamma \), the damping term, goes to zero. Because the angular frequency 
of the freely running damped oscillator is given by \( \omega_d = (\omega_0^2 - \gamma^2)^{1/2} \), we have
\[ \omega_\ast^2 = \omega_d^2 - \gamma^2 \]  
(3.6.11)

When the damping is weak, and only under this condition, the resonant frequency \( \omega_\ast \), 
the freely running, damped oscillator frequency \( \omega_d \), and the natural frequency \( \omega_0 \) of the 
undamped oscillator are essentially identical.

At the extreme of strong damping, no amplitude resonance occurs if \( \gamma > \omega_0/\sqrt{2} \), 
because the amplitude then becomes a monotonically decreasing function of \( \omega \). To see 
this, consider the limiting case \( \gamma^2 = \omega_0^2/2 \). Equation 3.6.9 then gives
\[ A(\omega) = \frac{F_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 2\omega_0^2 \omega^2}} = \frac{F_0/\sqrt{2}}{\left(\omega_0^2 + \omega^2\right)^{1/2}} \]  
(3.6.12)

which clearly decreases with increasing values of \( \omega \), starting with \( \omega = 0 \).
A seismograph may be modeled as a mass suspended by springs and a dashpot from a platform attached to the Earth (Figure 3.6.3). Oscillations of the Earth are passed through the platform to the suspended mass, which has a "pointer" to record its displacement relative to the platform. The dashpot provides a damping force. Ideally, the displacement $A$ of the mass relative to the platform should closely mimic the displacement of the Earth $D$. Find the equation of motion of the mass $m$ and choose parameters $\omega_0$ and $\gamma$ to ensure that $A$ lies within 10% of $D$. Assume during a ground tremor that the Earth oscillates with simple harmonic motion at $f = 10$ Hz.

Solution:

First we calculate the equation of motion of the mass $m$. Suppose the platform moves downward a distance $z$ relative to its initial position and that $m$ moves downward to a position $y$ relative to the platform. The plunger in the dashpot is moving downward with
speed \( z \) while the pot containing the damping fluid is moving downward with speed \( \dot{y} + z \); therefore, the retarding, damping force is given by \( c(ij) \). If \( l \) is the natural length of the spring, then

\[
F = mg - c(ij) - k(y - l) = m(\ddot{y} + \ddot{z})
\]

We let \( y = x + mg/k + l \), so that \( x \) is the displacement of the mass from its equilibrium position (see Figure 3.2.5), and, in terms of \( x \), the equation of motion becomes

\[
m\ddot{x} + c\dot{x} + kx = -m\ddot{z}
\]

During the tremor, as the platform oscillates with simple harmonic motion of amplitude \( D \) and angular frequency \( \omega = 2\pi f \), we have \( z = D e^{iwt} \). Thus,

\[
m\ddot{x} + c\dot{x} + kx = mD\omega^2 e^{iwt}
\]

Comparing with Equation 3.6.4, and associating \( F_0/m \) with \( D\omega^2 \), the solution for the amplitude of oscillation given by Equation 3.6.9 can be expressed here as

\[
A = D\omega^2 \left[ (\omega_0^2 - \omega^2)^2 + 4\gamma^2 \omega^2 \right]^{1/2}
\]

Dividing numerator and denominator by \( \omega^2 \) we obtain

\[
A = D\left[ \frac{\omega_0^2 - 1}{\omega^2} + 4\gamma^2 \right]^{1/2}
\]

Expanding the term in the denominator gives

\[
A = D\left[ 1 + \frac{\omega_0^2}{\omega^2} + \frac{2}{\omega^2} 2\gamma^2 \omega_0^2 \right]^{1/2}
\]
We can insure that \( A = D \) for reasonable values of \( \omega \) by setting \( 2\gamma^2 - \omega^2_0 = 0 \) and \( \omega_0/\omega < 0 \).
For example, for a fractional difference between \( A \) and \( D \) of 10\%, we require that
\[
\frac{D - A}{D} = 1 - \left(1 + \frac{\omega^2_0}{\omega^2}\right)^{-1/2} = \frac{1}{3} \frac{\omega^4_0}{\omega^4} < \frac{1}{10} \quad \text{or} \quad \omega_0 < 0.84\omega
\]
This means that the free-running frequency of the oscillator is
\[ f_0 = \omega_0/2\pi \leq 8 \text{ Hz} \]
The damping parameter should be
\[ \gamma = \omega_0/\sqrt{2} = 36. \]
Typically, this requires the use of "soft" springs and a heavy mass.

**Amplitude of Oscillation at the Resonance Peak**

The steady-state amplitude at the resonant frequency, which we call \( A_{\text{max}} \), is obtained from Equations 3.6.9 and 3.6.10. The result is
\[
A_{\text{max}} = \frac{F_0/m}{2\gamma\sqrt{\omega^2_0 - \gamma^2}}
\]
(3.6.13a)

In the case of weak damping, we can neglect \( \gamma^2 \) and write
\[
A_{\text{max}} = \frac{F_0}{2\gamma m\omega_0}
\]
(3.6.13b)

Thus, the amplitude of the induced oscillation at the resonant condition becomes very large if the damping factor \( \gamma \) is very small, and conversely. In mechanical systems large resonant amplitudes may or may not be desirable. In the case of electric motors, for example, rubber or spring mounts are used to minimize the transmission of vibration. The stiffness of these mounts is chosen so as to ensure that the resulting resonant frequency is far from the running frequency of the motor.

**Sharpness of the Resonance: Quality Factor**

The sharpness of the resonance peak is frequently of interest. Let us consider the case of weak damping \( \gamma \ll \omega_0 \). Then, in the expression for steady-state amplitude (Equation 3.6.9), we can make the following substitutions:
\[
\omega^2_0 - \omega^2 = (\omega_0 + \omega)(\omega_0 - \omega)
= 2\omega_0(\omega_0 - \omega)
\]
(3.6.14a)
\[
4\gamma^2\omega^2 = 4\gamma^2\omega_0^2
\]
(3.6.14b)
These, together with the expression for $A_{\text{max}}$, allow us to write the amplitude equation in the following approximate form:

$$A(\omega) = \frac{A_{\text{max}} \gamma}{\sqrt{(\omega_0 - \omega)^2 + \gamma^2}}$$  \hfill (3.6.15)

The above equation shows that when $|\omega_0 - \omega| = \gamma$ or, equivalently, if

$$\omega = \omega_0 \pm \gamma$$  \hfill (3.6.16)

then

$$A^2 = \frac{1}{2} A_{\text{max}}^2$$  \hfill (3.6.17)

This means that $\gamma$ is a measure of the width of the resonance curve. Thus, $2\gamma$ is the frequency difference between the points for which the energy is down by a factor of $e^2$ from the energy at resonance, because the energy is proportional to $A^2$.

The quality factor $Q$ defined in Equation 3.4.24, which characterizes the rate of energy loss in the undriven, damped harmonic oscillator, also characterizes the sharpness of the resonance peak for the driven oscillator. In the case of weak damping, $Q$ can be expressed as

$$Q = \frac{\omega_d}{2\gamma} = \frac{\omega_0}{2\gamma}$$  \hfill (3.6.18)

Thus, the total width $\Delta \omega$ at the half-energy points is approximately

$$\Delta \omega = 2\gamma = \frac{\omega_0}{Q}$$  \hfill (3.6.19a)

or, because $\omega = 2\pi f$,

$$\frac{\Delta \omega}{\omega_0} = \frac{\Delta f}{f_0} = \frac{1}{Q}$$  \hfill (3.6.19b)

giving the fractional width of the resonance peak.

This last expression for $Q$, so innocuous-looking, represents a key feature of feedback and control in electrical systems. Many electrical systems require the existence of a well-defined and precisely maintained frequency. High $Q$ (of order $10^5$) quartz oscillators, vibrating at their resonant frequency, are commonly employed as the control element in feedback circuits to provide frequency stability. A high $Q$ results in a sharp resonance. If the frequency of the circuit under control by the quartz oscillator starts to wander or drift by some amount $\delta f$ away from the resonance peak, feedback circuitry, exploiting the sharpness of the resonance, drives the circuit vigorously back toward the resonant frequency. The higher the $Q$ of the oscillator and, thus, the narrower $\delta f$, the more stable the output of the frequency of the circuit.
The Phase Difference $\phi$

Equation 3.6.8 gives the difference in phase $\phi$ between the applied driving force and the steady-state response:

$$\phi = \tan^{-1}\left[\frac{2\gamma\omega}{(\omega_0^2 - \omega^2)}\right]$$  \hspace{1cm} (3.6.20)

The phase difference is plotted in Figure 3.6.2(b). We saw that for the driven, undamped oscillator, $\phi$ was $0^\circ$ for $\omega < \omega_0$ and $180^\circ$ for $\omega > \omega_0$. These values are the low- and high-frequency limits of the real motion. Furthermore, $\phi$ changed discontinuously at $\omega = \omega_0$. This, too, is an idealization of the real motion where the transition between the two limits is smooth, although for very small damping it is quite abrupt, changing essentially from one limit to the other as $\omega$ passes through a region within $\pm\gamma$ about $\omega_0$.

At low driving frequencies $\omega \ll \omega_0$, we see that $\phi \to 0$ and the response is nearly in phase with the driving force. That this is reasonable can be seen upon examination of the amplitude of the oscillation (Equation 3.6.9). In the low-frequency limit, it becomes

$$A(\omega \to 0) = \frac{F_0}{2\gamma(-\omega_0^2)} = \frac{F_0}{k/m} = \frac{F_0}{k}$$ \hspace{1cm} (3.6.21)

In other words, just as we claimed during our preliminary discussion of the driven oscillator, the spring, and not the mass or the friction, controls the response; the mass is slowly pushed back and forth by a force acting against the retarding force of the spring.

At resonance the response can be enormous. Physically, how can this be? Perhaps some insight can be gained by thinking about pushing a child on a swing. How is it done? Clearly, anyone who has experience pushing a swing does not stand behind the child and push when the swing is on the backswing. One pushes in the same direction the swing is moving, essentially in phase with its velocity, regardless of its position. To push a small child, we usually stand somewhat to the side and give a very small shove forward as the swing passes through the equilibrium position, when its speed is a maximum and the displacement is zero! In fact, this is the optimum way to achieve a resonance condition; a rather gentle force, judiciously applied, can lead to a large amplitude of oscillation. The maximum amplitude at resonance is given by Equation 3.6.13a and, in the case of weak damping, by Equation 3.6.13b, $A_{\text{max}} = F_0/2\gamma m\omega_0$. But from the expression above for the amplitude as $\omega \to 0$, we have $A(\omega \to 0) = F_0/m\omega_0^2$. Hence, the ratio is

$$\frac{A_{\text{max}}}{A(\omega \to 0)} = \frac{F_0/(2\gamma m\omega_0)}{F_0/(m\omega_0^2)} = \frac{\omega_0}{2\gamma} = \omega_0 \tau = 2\pi$$ \hspace{1cm} (3.6.22)

The result is simply the $Q$ of the oscillator. Imagine what would happen to the child on the swing if there were no frictional losses! We would continue to pump little bits of energy into the swing on a cycle-by-cycle basis, and with no energy loss per cycle, the amplitude would soon grow to a catastrophic dimension.

Now let us look at the phase difference. At $\omega = \omega_0$, $\phi = \pi/2$. Hence, the displacement "lags," or is behind, the driving force by $90^\circ$. In view of the foregoing discussion, this should make sense. The optimum time to dump energy into the oscillator is when it swings
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through zero at maximum velocity, that is, when the power input $F \cdot v$ is a maximum. For example, the real part of Equation 3.6.5 gives the displacement of the oscillator:

$$x(t) = A(\omega) \Re(e^{i(\omega t - \phi)}) = A(\omega) \cos(\omega t - \phi)$$  \hspace{1cm} (3.6.23)

and at resonance, for small damping, this becomes

$$x(t) = A(\omega_0) \cos(\omega_0 t - \pi/2)$$ \hspace{1cm} (3.6.24)

The velocity, in general, is

$$\dot{x}(t) = -\omega A(\omega) \sin(\omega t - \phi)$$  \hspace{1cm} (3.6.25)

which at resonance becomes

$$\dot{x}(t) = \omega_0 A(\omega_0) \cos \omega_0 t$$  \hspace{1cm} (3.6.26)

Because the driving force at resonance is given by

$$F = F_0 \Re(e^{i\omega_0 t}) = F_0 \cos \omega_0 t$$  \hspace{1cm} (3.6.27)

we can see that the driving force is indeed in phase with the velocity of the oscillator, or $90^\circ$ ahead of the displacement.

Finally, for large values $\omega \gg \omega_0$, $\phi \rightarrow \pi$, and the displacement is $180^\circ$ out of phase with the driving force. The amplitude of the displacement becomes

$$A(\omega \gg \omega_0) = \frac{F_0}{m\omega_0^2}$$  \hspace{1cm} (3.6.28)

In this case, the amplitude falls off as $1/\omega^2$. The mass responds essentially like a free object, being rapidly shaken back and forth by the applied force. The main effect of the spring is to cause the displacement to lag behind the driving force by $180^\circ$.

**Electrical–Mechanical Analogs**

When an electric current flows in a circuit comprising inductive, capacitative, and resistive elements, there is a precise analogy with a moving mechanical system of masses and springs with frictional forces of the type studied previously. Thus, if a current $i = dq/dt$ ($q$ being the charge) flows through an inductance $L$, the potential difference across the inductance is $Lq$, and the stored energy is $\frac{1}{2}Lq^2$. Hence, inductance and charge are analogous to mass and displacement, respectively, and potential difference is analogous to force. Similarly, if a capacitance $C$ carries a charge $q$, the potential difference is $C^{-1}q$, and the stored energy is $\frac{1}{2}C^{-1}q^2$. Consequently, we see that the reciprocal of $C$ is analogous to the stiffness constant of a spring. Finally, for an electric current $i$ flowing through a resistance $R$, the potential difference is $iR = qR$, and the rate of energy dissipation is $i^2R = \dot{q}^2R$ in analogy with the quantity $c\dot{x}^2$ for a mechanical system. Table 3.6.1 summarizes the situation.
### TABLE 3.6.1

<table>
<thead>
<tr>
<th>Mechanical</th>
<th>Electrical</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>Displacement</td>
</tr>
<tr>
<td>$\dot{x}$</td>
<td>Velocity</td>
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<tr>
<td>$m$</td>
<td>Mass</td>
</tr>
<tr>
<td>$k$</td>
<td>Stiffness</td>
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<tr>
<td>$c$</td>
<td>Damping resistance</td>
</tr>
<tr>
<td>$F$</td>
<td>Force</td>
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</tbody>
</table>

### EXAMPLE 3.6.2

The exponential damping factor $\gamma$ of a spring suspension system is one-tenth the critical value. If the undamped frequency is $\omega_0$, find (a) the resonant frequency, (b) the quality factor, (c) the phase angle $\phi$ when the system is driven at a frequency $\omega = \omega_0/2$, and (d) the steady-state amplitude at this frequency.

**Solution:**

(a) We have $\gamma = \gamma_{\text{crit}}/10 = \omega_0/10$, from Equation 3.4.7, so from Equation 3.6.10,

$$\omega_r = \left[\omega_0^2 - 2(\omega_0/10)^2\right]^{1/2} = \omega_0(0.98)^{1/2} = 0.99\omega_0$$

(b) The system can be regarded as weakly damped, so, from Equation 3.6.18,

$$Q = \frac{\omega_0}{2\gamma} = \frac{\omega_0}{2(\omega_0/10)} = 5$$

(c) From Equation 3.6.8 we have

$$\phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2 - \omega^2}\right) = \tan^{-1}\left[\frac{2(\omega_0/10)(\omega_0/2)}{\omega_0^2 - (\omega_0/2)^2}\right]$$

$$= \tan^{-1} 0.133 = 7.6^\circ$$

(d) From Equation 3.6.9 we first calculate the value of the resonance denominator:

$$D(\omega = \omega_0/2) = \left[\left(\omega_0^2 - \omega_0^2/4\right)^2 + 4(\omega_0/10)^2(\omega_0/2)^2\right]^{1/2}$$

$$= \left[(9/16) + (1/100)\right]^{1/2} \omega_0^2 = 0.7566\omega_0^2$$

From this, the amplitude is

$$A(\omega = \omega_0/2) = \frac{F_0/m}{0.7566\omega_0^2} = 1.322 \frac{F_0}{m\omega_0^2}$$

Notice that the factor $\left(F_0/m\omega_0^2\right) = F_0/k$ is the steady-state amplitude for zero driving frequency.