Simple Harmonic Motion as the Projection of a Rotating Vector

Imagine a vector $\mathbf{A}$ rotating at a constant angular velocity $\omega_0$. Let this vector denote the position of a point $P$ in uniform circular motion. The projection of the vector onto a line (which we call the $x$-axis) in the same plane as the circle traces out simple harmonic motion. Suppose the vector $\mathbf{A}$ makes an angle $\theta$ with the $x$-axis at some time $t$, as shown in Figure 3.2.4. Because $\dot{\theta} = \omega_0$, the angle $\theta$ increases with time according to

$$\theta = \omega_0 t + \theta_0$$  \hspace{1cm} (3.2.14)

where $\theta_0$ is the value of $\theta$ at $t = 0$. The projection of $P$ onto the $x$-axis is given by

$$x = A \cos \theta = A \cos(\omega_0 t + \theta_0)$$  \hspace{1cm} (3.2.15)

This point oscillates in simple harmonic motion as $P$ goes around the circle in uniform angular motion.

Our picture describes $x$ as a cosine function of $t$. We can show the equivalence of this expression to the sine function given by Equation 3.2.5 by measuring angles to the vector $\mathbf{A}$ from the $y$-axis, instead of the $x$-axis as shown in Figure 3.2.4. If we do this, the projection of $\mathbf{A}$ onto the $x$-axis is given by

$$x = A \sin(\omega_0 t + \phi_0)$$  \hspace{1cm} (3.2.16)

We can see this equivalence in another way. We set the phase difference between $\phi_0$ and $\theta_0$ to $\pi/2$ and then substitute into the above equation, obtaining

$$\phi_0 - \theta_0 = \frac{\pi}{2}$$  \hspace{1cm} (3.2.17a)

$$\cos(\omega_0 t + \theta_0) = \cos(\omega_0 t + \phi_0 - \frac{\pi}{2})$$

$$= \sin(\omega_0 t + \phi_0)$$  \hspace{1cm} (3.2.17b)

Figure 3.2.4  Simple harmonic motion as a projection of uniform circular motion.
We now see that simple harmonic motion can be described equally well by a sine function or a cosine function. The one we choose is largely a matter of taste; it depends upon our choice of initial phase angle to within an arbitrary constant.

You might guess from the above commentary that we could use a sum of sine and cosine functions to represent the general solution for harmonic motion. For example, we can convert the sine solution of Equation 3.2.5 directly to such a form, using the trigonometric identity for the sine of a sum of angles:

\[ x(t) = A \sin(\omega_0 t + \phi_0) = A \sin \phi_0 \cos \omega_0 t + A \cos \phi_0 \sin \omega_0 t \]

\[ = C \cos \omega_0 t + D \sin \omega_0 t \]

(3.2.18)

Neither A nor \( \phi_0 \) appears explicitly in the solution. They are there implicitly; that is,

\[ \tan \phi_0 = \frac{C}{D} \]

\[ A^2 = C^2 + D^2 \]

(3.2.19)

There are occasions when this form may be the preferred one.

**Effect of a Constant External Force on a Harmonic Oscillator**

Suppose the same spring shown in Figure 3.2.1 is held in a vertical position, supporting the same mass \( m \) (Fig. 3.2.5). The total force acting is now given by adding the weight \( mg \) to the restoring force,

\[ F = -k(x - x_e) + mg \]

(3.2.20)

where the positive direction is down. This equation could be written \( F = -kx + mg \) by defining \( x \) to be \( X - x_e \), as previously. However, it is more convenient to define the variable \( x \) in a different way, namely, as the displacement from the new equilibrium position.

\[ x' = x - \frac{mg}{k} \]

(3.2.21)

\[ x_e = x_e + \frac{mg}{k} \]

\[ X_e = X_e + \frac{mg}{k} \]

**Figure 3.2.5** The vertical case for the harmonic oscillator.
3.2 Linear Restoring Force: Harmonic Motion

3.2.10 obtained by setting $F = 0$ in Equation 3.2.20: $0 = -k(X'_e - X_e) + mg$, which gives $X'_e = X_e + mg/k$. We now define the displacement as

$$x = X - X'_e = X - X_e - \frac{mg}{k} \quad (3.2.21)$$

Putting this into Equation 3.2.20 gives, after a very little algebra,

$$F = -kx \quad (3.2.22)$$

so the differential equation of motion is again

$$m\ddot{x} + kx = 0 \quad (3.2.23)$$

and our solution in terms of our newly defined $x$ is identical to that of the horizontal case. It should now be evident that any constant external force applied to a harmonic oscillator merely shifts the equilibrium position. The equation of motion remains unchanged if we measure the displacement $x$ from the new equilibrium position.

**Example 3.2.1**

When a light spring supports a block of mass $m$ in a vertical position, the spring is found to stretch by an amount $D_1$ over its unstretched length. If the block is furthermore pulled downward a distance $D_2$ from the equilibrium position and released—say, at time $t = 0$—find (a) the resulting motion, (b) the velocity of the block when it passes back upward through the equilibrium position, and (c) the acceleration of the block at the top of its oscillatory motion.

**Solution:**

First, for the equilibrium position we have

$$F_x = 0 = -kD_1 + mg$$

where $x$ is chosen positive downward. This gives us the value of the stiffness constant:

$$k = \frac{mg}{D_1}$$

From this we can find the angular frequency of oscillation:

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{D_1}}$$

We shall express the motion in the form $x(t) = A \cos \omega_0 t + B \sin \omega_0 t$. Then

$$\dot{x} = -A\omega_0 \sin \omega_0 t + B\omega_0 \cos \omega_0 t.$$ 

From the initial conditions we find

$$x_0 = D_2 = A \quad \dot{x}_0 = 0 = B\omega_0 \quad B = 0$$
The motion is, therefore, given by

\[ x(t) = D_2 \cos \left( \frac{g}{D_1} t \right) \]

in terms of the given quantities. Note that the mass \( m \) does not appear in the final expression. The velocity is then

\[ \dot{x}(t) = -D_2 \frac{g}{D_1} \sin \left( \frac{g}{D_1} t \right) \]

and the acceleration

\[ \ddot{x}(t) = -D_2 \frac{g^2}{D_1^2} \cos \left( \frac{g}{D_1} t \right) \]

As the block passes upward through the equilibrium position, the argument of the sine term is \( \pi/2 \) (one-quarter period), so

\[ \dot{x} = -D_2 \frac{g}{D_1} \quad \text{(center)} \]

At the top of the swing the argument of the cosine term is \( \pi \) (one-half period), which gives

\[ \ddot{x} = D_2 \frac{g}{D_1} \quad \text{(top)} \]

In the case \( D_1 = D_2 \), the downward acceleration at the top of the swing is just \( g \). This means that the block, at that particular instant, is in free fall; that is, the spring is exerting zero force on the block.

**EXAMPLE 3.2.2**

**The Simple Pendulum**

The so-called simple pendulum consists of a small plumb bob of mass \( m \) swinging at the end of a light, inextensible string of length \( l \), Figure 3.2.6. The motion is along a circular arc defined by the angle \( \theta \), as shown. The restoring force is the component of the weight \( mg \) acting in the direction of increasing \( \theta \) along the path of motion: \( F_\theta = -mg \sin \theta \). If we treat the bob as a particle, the differential equation of motion is, therefore,

\[ m\ddot{s} = -mg \sin \theta \]

Now \( s = l\theta \), and, for small \( \theta \), \( \sin \theta = \theta \) to a fair approximation. So, after canceling the \( m \)'s and rearranging terms, we can write the differential equation of motion in terms of either \( \theta \) or \( s \) as follows:

\[ \ddot{\theta} + \frac{g}{l} \theta = 0 \quad \ddot{s} + \frac{g}{l} s = 0 \]
Although the motion is along a curved path rather than a straight line, the differential equation is mathematically identical to that of the linear harmonic oscillator, Equation 3.2.4b, with the quantity $g/l$ replacing $km$. Thus, to the extent that the approximation $\sin \theta = \theta$ is valid, we can conclude that the motion is simple harmonic with angular frequency

$$\omega_0 = \sqrt{\frac{g}{l}}$$

and period

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{l}{g}}$$

This formula gives a period of very nearly 2 s, or a half-period of 1 s, when the length $l$ is 1 m. More accurately, for a half-period of 1 s, known as the "seconds pendulum," the precise length is obtained by setting $T_0 = 2$ s and solving for $l$. This gives $l = g/\pi^2$ numerically, when $g$ is expressed in m/s$^2$. At sea level at a latitude of 45°, the value of the acceleration of gravity is $g = 9.8062$ m/s$^2$. Accordingly, the length of a seconds pendulum at that location is $9.8062/9.8696 = 0.9936$ m.

### 3.3 Energy Considerations in Harmonic Motion

Consider a particle under the action of a linear restoring force $F_x = -kx$. Let us calculate the work done by an external force $F_{ext}$ in moving the particle from the equilibrium position $(x = 0)$ to some position $x$. Assume that we move the particle very slowly so that it does not gain any kinetic energy; that is, the applied external force is barely greater in magnitude than the restoring force $-kx$; hence, $F_{ext} = -F_x = kx$, so

$$W = \int_0^x F_{ext} \, dx = \int_0^x kx \, dx = \frac{k}{2} x^2$$

(3.3.1)
In the case of a spring obeying Hooke's law, the work is stored in the spring as potential energy: \( W = V(x) \), where

\[
V(x) = \frac{1}{2} kx^2
\]

(3.3.2)

Thus, \( F_x = -dV/dx = -kx \), as required by the definition of \( V \). The total energy, when the particle is undergoing harmonic motion, is given by the sum of the kinetic and potential energies, namely,

\[
E = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2
\]

(3.3.3)

This equation epitomizes the harmonic oscillator in a rather fundamental way: The kinetic energy is quadratic in the velocity variable, and the potential energy is quadratic in the displacement variable. The total energy is constant if there are no other forces except the restoring force acting on the particle.

The motion of the particle can be found by starting with the energy equation (3.3.3). Solving for the velocity gives

\[
\dot{x} = \pm \left( \frac{2E}{m} - \frac{kx^2}{m} \right)^{1/2}
\]

(3.3.4)

which can be integrated to give \( t \) as a function of \( x \) as follows:

\[
t = \int \pm \left( \frac{2E}{m} - \frac{kx^2}{m} \right)^{1/2} dx = \mp \left( \frac{m}{k} \right)^{1/2} \cos^{-1}(x/A) + C
\]

(3.3.5)

in which \( C \) is a constant of integration and \( A \) is the amplitude given by

\[
A = \left( \frac{2E}{k} \right)^{1/2}
\]

(3.3.6)

Upon solving the integrated equation for \( x \) as a function of \( t \), we find the same relationship as in the preceding section, with the addition that we now have an explicit value for the amplitude. We can also obtain the amplitude directly from the energy equation (3.3.3) by finding the turning points of the motion where \( \dot{x} = 0 \): The value of \( x \) must lie between \( \pm A \) in order for \( \dot{x} \) to be real. This is illustrated in Figure 3.3.1.

**Figure 3.3.1** Graph of the parabolic potential energy function of the harmonic oscillator. The turning points defining the amplitude are indicated for two different values of the total energy.
We also see from the energy equation that the maximum value of the speed, which we call \( v_{\text{max}} \), occurs at \( x = 0 \). Accordingly, we can write

\[
E = \frac{1}{2} m v_{\text{max}}^2 = \frac{1}{2} k A^2 \tag{3.3.7}
\]

As the particle oscillates, the kinetic and potential energies continually change. The constant total energy is entirely in the form of kinetic energy at the center, where \( x = 0 \) and \( \dot{x} = \pm v_{\text{max}} \), and it is all potential energy at the extrema, where \( \dot{x} = 0 \) and \( x = \pm A \).

**Example 3.3.1**

**The Energy Function of the Simple Pendulum**

The potential energy of the simple pendulum (Fig. 3.2.6) is given by the expression

\[
V = mgh
\]

where \( h \) is the vertical distance from the reference level (which we choose to be the level of the equilibrium position). For a displacement through an angle \( \theta \) (Fig. 3.2.6), we see that \( h = l - l \cos \theta \), so

\[
V(\theta) = mgl(1 - \cos \theta)
\]

Now the series expansion for the cosine is \( \cos \theta = 1 - \theta^2/2! + \theta^4/4! - \cdots \), so for small \( \theta \) we have approximately \( \cos \theta = 1 - \theta^2/2 \). This gives

\[
V(\theta) = \frac{1}{2} mgl \theta^2
\]

or, equivalently, because \( s = l \theta \),

\[
V(s) = \frac{1}{2} \frac{mg}{l} s^2
\]

Thus, to a first approximation, the potential energy function is quadratic in the displacement variable. In terms of \( s \), the total energy is given by

\[
E = \frac{1}{2} ms^2 + \frac{1}{2} \frac{mg}{l} s^2
\]

in accordance with the general statement concerning the energy of the harmonic oscillator discussed above.

**Example 3.3.2**

Calculate the average kinetic, potential, and total energies of the harmonic oscillator.

(Here we use the symbol \( K \) for kinetic energy and \( T_0 \) for the period of the motion.)

**Solution:**

\[
\langle K \rangle = \frac{1}{T_0} \int_0^{T_0} K(t) \, dt = \frac{1}{T_0} \int_0^{T_0} \frac{1}{2} m \dot{x}^2 \, dt
\]
CHAPTER 3 Oscillations

but

\[ x = A \sin(\omega_0 t + \phi_0) \]
\[ \dot{x} = \omega_0 A \cos(\omega_0 t + \phi_0) \]

Setting \( \phi_0 = 0 \) and letting \( \omega = \omega_0 t = (2\pi/T_0) \cdot t \), we obtain

\[
\langle K \rangle = \frac{1}{T_0} \left[ \int_0^{T_0} \frac{1}{2} m \omega_0^2 A^2 \cos^2(\omega_0 t) \, dt \right]
\]
\[
= \frac{1}{2\pi} \left[ \int_0^{2\pi} \frac{1}{2} m \omega_0^2 A^2 \cos^2 u \, du \right]
\]

We can make use of the fact that

\[
\frac{1}{2\pi} \int_0^{2\pi} \sin^2 u \, du = \frac{1}{2\pi} \int_0^{2\pi} \cos^2 u \, du = 1
\]

to obtain

\[
\frac{1}{2\pi} \int_0^{2\pi} \cos^2 u \, du = \frac{1}{2}
\]

because the areas under the \( \cos^2 \) and \( \sin^2 \) terms throughout one cycle are identical. Thus,

\[
\langle K \rangle = \frac{1}{4} m \omega_0^2 A^2
\]

The calculation of the average potential energy proceeds along similar lines.

\[
V = \frac{1}{2} k x^2 = \frac{1}{2} k A^2 \sin^2 \omega_0 t
\]
\[
\langle V \rangle = \frac{1}{2} k A^2 \frac{1}{T_0} \int_0^{T_0} \sin^2 \omega_0 t \, dt
\]
\[
= \frac{1}{2} k A^2 \frac{1}{2\pi} \int_0^{2\pi} \sin^2 u \, du
\]
\[
= \frac{1}{4} k A^2
\]

Now, because \( k/m = \omega_0^2 \) or \( k = m \omega_0^2 \), we obtain

\[
\langle V \rangle = \frac{1}{4} k A^2 = \frac{1}{4} m \omega_0^2 A^2 = \langle K \rangle
\]
\[
\langle E \rangle = \langle K \rangle + \langle V \rangle = \frac{1}{2} m \omega_0^2 A^2 = \frac{1}{2} k A^2 = E
\]

The average kinetic energies and potential energies are equal; therefore, the average energy of the oscillator is equal to its total instantaneous energy.

### 3.4 Damped Harmonic Motion

The foregoing analysis of the harmonic oscillator is somewhat idealized in that we have failed to take into account frictional forces. These are always present in a mechanical system to some extent. Analogously, there is always a certain amount of resistance in an electrical circuit. For a specific model, let us consider an object of mass \( m \) that is supported by