CHAPTER 3 Oscillations

3.4 Damped Harmonic Motion

The foregoing analysis of the harmonic oscillator is somewhat idealized in that we have failed to take into account frictional forces. These are always present in a mechanical system to some extent. Analogously, there is always a certain amount of resistance in an electrical circuit. For a specific model, let us consider an object of mass $m$ that is supported by
a light spring of stiffness $k$. We assume that there is a viscous retarding force that is a linear function of the velocity, such as is produced by air drag at low speeds. The forces are indicated in Figure 3.4.1.

If $x$ is the displacement from equilibrium, then the restoring force is $-kx$, and the retarding force is $-c\dot{x}$, where $c$ is a constant of proportionality. The differential equation of motion is, therefore, $m\ddot{x} = -kx - c\dot{x}$, or

$$m\ddot{x} + c\dot{x} + kx = 0 \quad (3.4.1)$$

As with the undamped case, we divide Equation 3.4.1 by $m$ to obtain

$$\ddot{x} + \frac{c}{m} \dot{x} + \frac{k}{m} x = 0 \quad (3.4.2)$$

If we substitute the damping factor $\gamma$, defined as

$$\gamma = \frac{c}{2m} \quad (3.4.3)$$

and $\omega_0^2 (= k/m)$ into Equation 3.4.2, it assumes the simpler form

$$\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = 0 \quad (3.4.4)$$

The presence of the velocity-dependent term $2\gamma \dot{x}$ complicates the problem; simple sine or cosine solutions do not work, as can be verified by trying them. We introduce a method of solution that works rather well for second-order differential equations with constant

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Figure 3.4.1 A model for the damped harmonic oscillator.

2Nonlinear drag is more realistic in many situations; however, the equations of motion are much more difficult to solve and are not treated here.
coefficients. Let $D$ be the differential operator $d/dt$. We "operate" on $x$ with a quadratic
function of $D$ chosen in such a way that we generate Equation 3.4.4:

$$[D^2 + 2\gamma D + \omega_0^2]x = 0$$ (3.4.5a)

We interpret this equation as an "operation" by the term in brackets on $x$. The operation
by $D^2$ means first operate on $x$ with $D$ and then operate on the result of that operation with
$D$ again. This procedure yields $\dddot{x}$, the first term in Equation 3.4.4. The operator equation
(Equation 3.4.5a) is, therefore, equivalent to the differential equation (Equation 3.4.4).

The simplification that we get by writing the equation this way arises when we factor the
operator term, using the binomial theorem, to obtain

$$[D + y - \sqrt{\gamma^2 - \omega_0^2}]D + y + \sqrt{\gamma^2 - \omega_0^2}x = 0$$ (3.4.5b)

The operation in Equation 3.4.5b is identical to that in Equation 3.4.5a, but we have
reduced the operation from second-order to a product of two first-order ones. Because
the order of operation is arbitrary, the general solution is a sum of solutions obtained by
setting the result of each first-order operation on $x$ equal to zero. Thus, we obtain

$$x(t) = A_1 e^{(\gamma-q)t} + A_2 e^{(\gamma+q)t}$$ (3.4.6)

where

$$q = \sqrt{\gamma^2 - \omega_0^2}$$ (3.4.7)

The student can verify that this is a solution by direct substitution into Equation 3.4.4.
A problem that we soon encounter, though, is that the above exponents may be real
or complex, because the factor $q$ could be imaginary. We see what this means in just a
minute.

There are three possible scenarios:

I. $q$ real $> 0$ \hspace{1cm} \textit{Overdamping}

II. $q$ real $= 0$ \hspace{1cm} \textit{Critical damping}

III. $q$ imaginary \hspace{1cm} \textit{Underdamping}

I. \textit{Overdamped}. Both exponents in Equation 3.4.6 are real. The constants $A_1$ and
$A_2$ are determined by the initial conditions. The motion is an exponential decay
with two different decay constants, $(\gamma - q)$ and $(\gamma + q)$. A mass, given some ini-
tial displacement and released from rest, returns slowly to equilibrium, pre-
vented from oscillating by the strong damping force. This situation is depicted
in Figure 3.4.2.

II. \textit{Critical damping}. Here $q = 0$. The two exponents in Equation 3.4.6 are each equal
to $\gamma$. The two constants $A_1$ and $A_2$ are no longer independent. Their sum forms a single
constant $A$. The solution degenerates to a single exponential decay function. A com-
pletely general solution requires two different functions and independent constants
to satisfy the boundary conditions specified by an initial position and velocity. To find
a solution with two independent constants, we return to Equation 3.4.5b:

$$(D + \gamma)(D + \gamma)x = 0$$ (3.4.8a)
3.4 Damped Harmonic Motion

Switching the order of operation does not work here, because the operators are the same. We have to carry out the entire operation on $x$ before setting the result to zero. To do this, we make the substitution $u = (D + \gamma)x$, which gives

$$ (D + \gamma)u = 0 $$

$$ u = Ae^{-\eta t} $$

Equating this to $(D + \gamma)x$, the final solution is obtained as follows:

$$ Ae^{-\eta t} = (D + \gamma)x $$

$$ A = e^{\eta t}(D + \gamma)x = D(\xi e^{\eta t}) $$

$$ x(t) = Ate^{-\eta t} + Be^{-\eta t} $$

The solution consists of two different functions, $te^{-\eta t}$ and $e^{-\eta t}$, and two constants of integration, $A$ and $B$, as required. As in case I, if a mass is released from rest after an initial displacement, the motion is nonoscillatory, returning asymptotically to equilibrium. This case is also shown in Figure 3.4.2. Critical damping is highly desirable in many systems, such as the mechanical suspension systems of motor vehicles.

III. Underdamping. If the constant $\gamma$ is small enough that $\gamma^2 - \omega_0^2 < 0$, the factor $q$ in Equation 3.4.7 is imaginary. A mass initially displaced and then released from rest oscillates, not unlike the situation described earlier for no damping force at all. The only difference is the presence of the real factor $-\gamma$ in the exponent of the solution that leads to the ultimate death of the oscillatory motion. Let us now reverse the factors under the square root sign in Equation 3.4.7 and write $q$ as $i\omega_d$. Thus,

$$ \omega_d = \sqrt{\omega_0^2 - \gamma^2} = \frac{k}{m} - \frac{c^2}{4m^2} $$

where $\omega_0$ and $\omega_d$ are the angular frequencies of the undamped and underdamped harmonic oscillators, respectively. We now rewrite the general solution represented by Equation 3.4.6 in terms of the factors described here,

$$ x(t) = C_1e^{-(\gamma - i\omega_d)t} + C_2e^{-(\gamma + i\omega_d)t} $$

$$ = e^{-\eta t}(C_1e^{i\omega_d t} + C_2e^{-i\omega_d t}) $$

Figure 3.4.2  Displacement versus time for critically damped and overdamped oscillators released from rest after an initial displacement.
where the constants of integration are $C_+$ and $C_-$. The solution contains a sum of imaginary exponentials. But the solution must be real—it is supposed to describe the real world! This reality demands that $C_+$ and $C_-$ be complex conjugates of each other, a condition that ultimately allows us to express the solution in terms of sines and/or cosines. Thus, taking the complex conjugate of Equation 3.4.11,

$$x^*(t) = e^{-\gamma t} (C_+ e^{-i\omega_d t} + C_- e^{+i\omega_d t}) = x(t) \quad (3.4.12a)$$

Because $x(t)$ is real, $x^*(t) = x(t)$, and, therefore,

$$\therefore \ C_+^* = C_- = C \quad (3.4.12b)$$

$$\therefore \ C_+ = C^* \quad (3.4.12b)$$

$$\therefore \ x(t) = e^{-\gamma t} (C^* e^{+i\omega_d t} + C e^{-i\omega_d t})$$

It looks as though we have a solution that now has only a single constant of integration. In fact, $C$ is a complex number. It is composed of two constants. We can express $C$ and $C^*$ in terms of two real constants, $A$ and $\theta_0$, in the following way.

$$C_- = C = \frac{A}{2} e^{-i\theta_0} \quad (3.4.13)$$

$$C_+ = C^* = \frac{A}{2} e^{+i\theta_0}$$

We soon see that $A$ is the maximum displacement and $\theta_0$ is the initial phase angle of the motion. Thus, Equation 3.4.12b becomes

$$x(t) = e^{-\gamma t} \left( \frac{A}{2} e^{+i(\omega_d t + \theta_0)} + \frac{A}{2} e^{-i(\omega_d t + \theta_0)} \right) \quad (3.4.14)$$

We now apply Euler's identity$^3$ to the above expressions, thus obtaining

$$\frac{A}{2} e^{+(\omega_d t + \theta_0)} = \frac{A}{2} \cos(\omega_d t + \theta_0) + i \frac{A}{2} \sin(\omega_d t + \theta_0)$$

$$\frac{A}{2} e^{-i(\omega_d t + \theta_0)} = \frac{A}{2} \cos(\omega_d t + \theta_0) - i \frac{A}{2} \sin(\omega_d t + \theta_0) \quad (3.4.15)$$

$$\therefore \ x(t) = e^{-\gamma t} (A \cos(\omega_d t + \theta_0))$$

Following our discussion in Section 3.2 concerning the rotating vector construct, we see that we can express the solution equally well as a sine function:

$$x(t) = e^{-\gamma t} (A \sin(\omega_d t + \phi_0)) \quad (3.4.16)$$

The constants $A$, $\theta_0$, and $\phi_0$ have the same interpretation as those of Section 3.2. In fact, we see that the solution for the underdamped oscillator is nearly identical to that of the undamped oscillator. There are two differences: (1) The presence of the real exponential factor $e^{-\gamma t}$ leads to a gradual death of the oscillations, and (2) the underdamped oscillator's angular frequency is $\omega_d$, not $\omega_0$, because of the presence of the damping force.

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$^3$Euler's identity relates imaginary exponentials to sines and cosines. It is given by the expression $e^{iu} = \cos u + i \sin u$. This equality is demonstrated in Appendix D.
The underdamped oscillator vibrates a little more slowly than does the undamped oscillator. The period of the underdamped oscillator is given by

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\left(\omega_0^2 - \gamma^2\right)^{1/2}}$$  \hspace{1cm} (3.4.17)

Figure 3.4.3 is a plot of the motion. Equation 3.4.15a shows that the two curves given by $x = A e^{-\gamma t}$ and $x = -A e^{-\gamma t}$ form an envelope of the curve of motion because the cosine factor takes on values between +1 and -1, including +1 and -1, at which points the curve of motion touches the envelope. Accordingly, the points of contact are separated by a time interval of one-half period, $T_d/2$. These points, however, are not quite the maxima and minima of the displacement. It is left to the student to show that the actual maxima and minima are also separated in time by the same amount. In one complete period the amplitude diminishes by a factor $e^{-\gamma T_d}$; also, in a time $\gamma^{-1} = 2m/c$ the amplitude decays by a factor $e^{-1} = 0.3679$.

In summary, our analysis of the freely running harmonic oscillator has shown that the presence of damping of the linear type causes the oscillator, given an initial motion, to eventually return to a state of rest at the equilibrium position. The return to equilibrium is either oscillatory or not, depending on the amount of damping. The critical condition, given by $\gamma = \omega_0$, characterizes the limiting case of the nonoscillatory mode of return.

**Energy Considerations**

The total energy of the damped harmonic oscillator is given by the sum of the kinetic and potential energies:

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$$  \hspace{1cm} (3.4.18)

This is constant for the undamped oscillator, as stated previously. Let us differentiate the above expression with respect to $t$:

$$\frac{dE}{dt} = m \ddot{x} + k x = (m \ddot{x} + k x) \dot{x}$$  \hspace{1cm} (3.4.19)
Now the differential equation of motion is \( m\ddot{x} + c\dot{x} + kx = 0 \), or \( m\ddot{x} + kx = -c\dot{x} \). Thus, we can write

\[
\frac{dE}{dt} = -c\dot{x}^2
\]

(3.4.20)

for the time rate of change of total energy. We see that it is given by the product of the damping force and the velocity. Because this is always either zero or negative, the total energy continually decreases and, like the amplitude, eventually becomes negligibly small. The energy is dissipated as frictional heat by virtue of the viscous resistance to the motion.

**Quality Factor**

The rate of energy loss of a weakly damped harmonic oscillator is best characterized by a single parameter \( Q \), called the quality factor of the oscillator. It is defined to be \( 2\pi \) times the energy stored in the oscillator divided by the energy lost in a single period of oscillation \( T_d \). If the oscillator is weakly damped, the energy lost per cycle is small and \( Q \) is, therefore, large. We calculate \( Q \) in terms of parameters already derived and show that this is true.

The average rate of energy dissipation for the damped oscillator is given by Equation 3.4.20, \( E = -c\dot{x}^2 \), so we need to calculate \( \dot{x} \). Equation 3.4.16 gives \( x(t) \):

\[
x = Ae^{-\gamma t} \sin(\omega_d t + \phi_0)
\]

(3.4.21a)

Differentiating it, we obtain

\[
\dot{x} = -Ae^{-\gamma t}(\gamma \sin(\omega_d t + \phi_0) - \omega_d \cos(\omega_d t + \phi_0))
\]

(3.4.21b)

The energy lost during a single cycle of period \( T_d = 2\pi/\omega_d \) is

\[
\Delta E = \int_0^{T_d} \dot{E} \, dt
\]

(3.4.22a)

If we change the variable of integration to \( \theta = \omega_d t + \phi_0 \), then \( dt = d\theta/\omega_d \) and the integral over the period \( T_d \) transforms to an integral, from \( \phi_0 \) to \( \phi_0 + 2\pi \). The value of the integral over a full cycle doesn’t depend on the initial phase \( \phi_0 \) of the motion, so, for the sake of simplicity, we drop it from the limits of integration:

\[
\Delta E = \frac{1}{\omega_d} \int_0^{2\pi} \dot{E} \, d\theta
\]

\[
= -\frac{cA^2}{\omega_d} \int_0^{2\pi} e^{-2\gamma\theta} \left[ \gamma^2 \sin^2 \theta - 2\gamma \omega_d \sin \theta \cos \theta + \omega_d^2 \cos^2 \theta \right] \, d\theta
\]

(3.4.22b)

Now we can extract the exponential factor \( e^{-2\gamma \theta} \) from inside the integral, because in the case of weak damping (\( \gamma \ll \omega_d \)) its value does not change very much during a single cycle of oscillation:

\[
\Delta E = \frac{-cA^2}{\omega_d} e^{-2\gamma T_d} \int_0^{2\pi} \left( \gamma^2 \sin^2 \theta - 2\gamma \omega_d \sin \theta \cos \theta + \omega_d^2 \cos^2 \theta \right) \, d\theta
\]

(3.4.22c)
The integral of both $\sin^2 \theta$ and $\cos^2 \theta$ over one cycle is $\pi$, while the integral of the $\sin \theta \cos \theta$ product vanishes. Thus, we have

$$\Delta E = \frac{-cA^2}{\omega_d} \pi e^{-2\gamma t} \left(\omega_0^2 + \omega_d^2\right) = -cA^2 e^{-2\gamma t} \omega_d^2 \left(\frac{\pi}{\omega_d}\right)$$

(3.4.22d)

where we have made use of the relations $\omega_0^2 = \omega_d^2 + \gamma^2$ and $\gamma = c/2m$. Now, if we identify the damping factor $\gamma$ with a time constant $\tau$, such that $\gamma = (2\tau)^{-1}$, we obtain for the magnitude of the energy loss in one cycle

$$\Delta E = \left(\frac{1}{2} m \omega_d^2 \omega_0^2 e^{-\tau t}\right) \frac{T_d}{\tau}$$

(3.4.22e)

where the energy stored in the oscillator (see Example 3.3.2) at any time $t$ is

$$E(t) = \frac{1}{2} m \omega_d^2 A^2 e^{-\omega_d t}$$

(3.4.23)

Clearly, the energy remaining in the oscillator during any cycle dies away exponentially with time constant $\tau$. We, therefore, see that the quality factor $Q$ is just $2\pi$ times the inverse of the ratios given in the expression above, or

$$Q = \frac{2\pi}{(T_d/\tau)} = \frac{2\pi \tau}{(2\pi/\omega_d)} = \omega_d \tau = \frac{\omega_d}{2\gamma}$$

(3.4.24)

For weak damping, the period of oscillation $T_d$ is much less than the time constant $\tau$, which characterizes the energy loss rate of the oscillator. $Q$ is large under such circumstances. Table 3.4.1 gives some values of $Q$ for several different kinds of oscillators.

<table>
<thead>
<tr>
<th>TABLE 3.4.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Earth (for earthquake)</td>
</tr>
<tr>
<td>Piano string</td>
</tr>
<tr>
<td>Crystal in digital watch</td>
</tr>
<tr>
<td>Microwave cavity</td>
</tr>
<tr>
<td>Excited atom</td>
</tr>
<tr>
<td>Neutron star</td>
</tr>
<tr>
<td>Excited Fe$^{57}$ nucleus</td>
</tr>
</tbody>
</table>

**EXAMPLE 3.4.1**

An automobile suspension system is critically damped, and its period of free oscillation with no damping is 1 s. If the system is initially displaced by an amount $x_0$ and released with zero initial velocity, find the displacement at $t = 1$ s.

**Solution:**

For critical damping we have $\gamma = c/2m = (k/m)^{1/2} = \omega_0 = 2\pi T_0$. Hence, $\gamma = 2\pi s^{-1}$ in our case, because $T_0 = 1$ s. Now the general expression for the displacement in the critically
damped case (Equation 3.4.9) is \( x(t) = (At + B)e^{-\gamma t} \), so, for \( t = 0, x_0 = B \). Differentiating, we have \( \ddot{x}(t) = (A - \gamma B - \gamma At)e^{-\gamma t} \), which gives \( \ddot{x}_0 = A - \gamma B = 0 \), so \( A = \gamma B = \gamma x_0 \) in our problem. Accordingly,

\[
x(t) = x_0(1 + \gamma t)e^{-\gamma t} = x_0(1 + 2\pi t)e^{-2\pi t}
\]

is the displacement as a function of time. For \( t = 1 \text{ s} \), we obtain

\[
x_0(1 + 2\pi)e^{-2\pi} = x_0(7.28)e^{-6.28} = 0.0136 x_0
\]

The system has practically returned to equilibrium.

**EXAMPLE 3.4.2**

The frequency of a damped harmonic oscillator is one-half the frequency of the same oscillator with no damping. Find the ratio of the maxima of successive oscillations.

**Solution:**

We have \( \omega_d = \frac{1}{2} \omega_0 = (\omega_0^2 - \gamma^2)^{1/2} \), which gives \( \omega_0^2/4 = \omega_0^2 - \gamma^2 \), so \( \gamma = \omega_0(3/4)^{1/2} \). Consequently,

\[
\gamma T_d = \omega_0(3/4)^{1/2} \left[ 2\pi/(\omega_0/2) \right] = 10.88
\]

Thus, the amplitude ratio is

\[
e^{-\gamma T_d} = e^{-10.88} = 0.00002
\]

This is a highly damped oscillator.

**EXAMPLE 3.4.3**

Given: The terminal speed of a baseball in free fall is 30 m/s. Assuming a linear air drag, calculate the effect of air resistance on a simple pendulum, using a baseball as the plumb bob.

**Solution:**

In Chapter 2 we found the terminal speed for the case of linear air drag to be given by \( v_t = mg/c_1 \), where \( c_1 \) is the linear drag coefficient. This gives

\[
\gamma = \frac{c_1}{2m} = \frac{(mg/v_t)}{2m} = \frac{g}{2v_t} = \frac{9.8 \text{ ms}^{-2}}{60 \text{ ms}^{-1}} = 0.163 \text{ s}^{-1}
\]

for the exponential damping constant. Consequently, the baseball pendulum’s amplitude drops off by a factor \( e^{-1} \) in a time \( \gamma^{-1} = 6.13 \text{ s} \). This is independent of the length of the pendulum. Earlier, in Example 3.2.2, we showed that the angular frequency of oscillation of the simple pendulum of length \( l \) is given by \( \omega_0 = (g/l)^{1/2} \) for small amplitude. Therefore, from Equation 3.4.17, the period of our pendulum is

\[
T_d = 2\pi \left( \omega_0^2 - \gamma^2 \right)^{-1/2} = 2\pi \left( \frac{g}{l} - 0.0265 \text{ s}^{-2} \right)^{-1/2}
\]
In particular, for a baseball "seconds pendulum" for which the half-period is 1 s in the absence of damping, we have $g/l = \pi^2$, so the half-period with damping in our case is

$$T_d = \frac{\pi}{2} (\pi^2 - 0.0265)^{1/2} \approx 1.00134 \text{ s}$$

Our solution somewhat exaggerates the effect of air resistance, because the drag function for a baseball is more nearly quadratic than linear in the velocity except at very low velocities, as discussed in Section 2.4.

**Example 3.4.4**

A spherical ball of radius 0.00265 m and mass $5 \times 10^{-4}$ kg is attached to a spring of force constant $k = 0.05$ N/m underwater. The mass is set to oscillate under the action of the spring. The coefficient of viscosity $\eta$ for water is $10^{-3}$ Ns/m². (a) Find the number of oscillations that the ball will execute in the time it takes for the amplitude of the oscillation to drop by a factor of 2 from its initial value. (b) Calculate the $Q$ of the oscillator.

**Solution:**

Stokes’ law for objects moving in a viscous medium can be used to find $c$, the constant of proportionality of the $x$ term, in the equation of motion (Equation 3.4.1) for the damped oscillator. The relationship is

$$c = 6\pi \eta r = 5 \cdot 10^{-5} \text{ Ns/m}$$

The energy of the oscillator dies away exponentially with time constant $\tau$, and the amplitude dies away as $A = A_0 e^{-\omega \tau}$. Thus,

$$A = \frac{1}{2} = e^{-\omega \tau} \quad \therefore \tau = \frac{\ln 2}{\omega}$$

Consequently, the number of oscillations during this time is

$$n = \frac{\omega \tau}{2\pi} = \frac{\omega \tau}{2\pi} = \frac{Q (\ln 2)/\pi}{2\pi}$$

Because $\omega^2 = k/m = 100 \text{ s}^{-2}$, $\tau = m/c = 10 \text{ s}$, and $\gamma = 1/2\tau = 0.05 \text{ s}^{-1}$, we obtain

$$Q = \left(\omega^2 - \gamma^2\right)^{1/2} \approx (100 - 0.0025)^{1/2} 10 = 100$$

$$n = \frac{Q (\ln 2) / \pi}{2\pi} = 22$$

If we had asked how many oscillations would occur in the time it takes for the amplitude to drop to $e^{-1/2}$, or about 0.606 times its initial value, the answer would have been $Q/2\pi$. Clearly $Q$ is a measure of the rate at which an oscillator loses energy.
A physical system in motion that does not dissipate energy remains in motion. One that dissipates energy eventually comes to rest. An oscillating or rotating system that does not dissipate energy repeats its configuration each cycle. One that dissipates energy never does. The evolution of such a physical system can be graphically illustrated by examining its motion in a special space called phase space, rather than real space. The phase space for a single particle whose motion is restricted to lie along a single spatial coordinate consists of all the possible points in a “plane” whose horizontal coordinate is its position \( x \) and whose vertical coordinate is its velocity \( \dot{x} \). Thus, the “position” of a particle on the phase-space plane is given by its “coordinates” \((x, \dot{x})\).

The future state of motion of such a particle is completely specified if its position and velocity are known simultaneously—say, its initial conditions \( x(t_0) \) and \( \dot{x}(t_0) \). We can, thus, picture the evolution of the motion of the particle from that point on by plotting its coordinates in phase space. Each point in such a plot can be thought of as a precursor for the next point. The trajectory of these points in phase space represents the complete time history of the particle.

### Simple Harmonic Oscillator: No Damping Force

The simple harmonic oscillator that we discuss in this section is an example of a particle whose motion is restricted to a single dimension. Let’s examine the phase-space motion of a simple harmonic oscillator that is not subject to any damping force. The solutions for its position and velocity as functions of time were given previously by Equations 3.2.5 and 3.2.12a:

\[
\begin{align*}
    x(t) &= A \sin(\omega_0 t + \phi_0) \\
    \dot{x}(t) &= \omega_0 A \cos(\omega_0 t + \phi_0)
\end{align*}
\]

Letting \( y = \dot{x} \) we eliminate \( t \) from these two parametric equations to find the equation of the trajectory of the oscillator in phase space:

\[
\frac{x^2(t) + \dot{x}^2(t)}{\omega_0^2} = A^2 \sin^2(\omega_0 t + \phi_0) + \cos^2(\omega_0 t + \phi_0) = A^2
\]

\[
\therefore \frac{x^2}{A^2} + \frac{\dot{x}^2}{\omega_0^2} = 1
\]

Equation 3.5.2 is the equation of an ellipse whose semimajor axis is \( A \) and whose semiminor axis is \( \omega_0 A \). Shown in Figure 3.5.1 are several phase-space trajectories for the harmonic oscillator. The trajectories differ only in the amplitude \( A \) of the oscillation.

Note that the phase-path trajectories never intersect. The existence of a point common to two different trajectories would imply that two different future motions could evolve.

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*Again, as noted in Chapter 2, sections in the text marked with an asterisk may be skipped with impunity.

4Strictly speaking, phase space is defined as the ensemble of points \((x, p)\) where \( x \) and \( p \) are the position and momentum of the particle. Because momentum is directly proportional to velocity, the space defined here is essentially a phase space.
from a single set of conditions \((x(t_1), \dot{x}(t_1))\) at some time \(t_1\). This cannot happen because, starting with specific values of \(x(t_1)\) and \(\dot{x}(t_1)\), Newton's laws of motion completely determine a unique future state of motion for the system.

Also note that the trajectories in this case form closed paths. In other words, the motion repeats itself, a consequence of the conservation of the total energy of the harmonic oscillator. In fact, the equation of the phase-space trajectory (Equation 3.5.2) is nothing more than a statement that the total energy is conserved. We can show this by substituting \(E = \frac{1}{2} kA^2\) and \(\omega_0^2 = k/m\) into Equation 3.5.2, obtaining

\[
\frac{x^2}{2E/k} + \frac{y^2}{2E/m} = 1
\]  \(\text{(3.5.3a)}\)

which is equivalent to (replacing \(y\) with \(\dot{x}\))

\[
\frac{1}{2} kx^2 + \frac{1}{2} m\dot{x}^2 = V + T = E
\]  \(\text{(3.5.3b)}\)

the energy equation (Equation 3.3.3) for the harmonic oscillator. Each closed phase-space trajectory, thus, corresponds to some definite, conserved total energy.

**Example 3.5.1**

Consider a particle of mass \(m\) subject to a force of strength \(+kx\), where \(x\) is the displacement of the particle from equilibrium. Calculate the phase space trajectories of the particle.

**Solution:**

The equation of motion of the particle is \(m\ddot{x} = kx\). Letting \(\omega^2 = k/m\) we have \(\ddot{x} - \omega^2 x = 0\). Letting \(y = x\) and \(y' = dy/dx\) we have \(y = x\) and \(y' = \omega^2 x\) or \(y dy = \omega^2 x dx\). The solution is \(y^2 - \omega^2 x^2 = C\) in which \(C\) is a constant of integration. The phase space trajectories are branches of a hyperbola whose asymptotes are \(y = \pm \omega x\). The resulting phase space plot is shown in Figure 3.5.2. The trajectories are open ended, radiating away from the origin, which is an unstable equilibrium point.
The Underdamped Harmonic Oscillator

The phase-space trajectories for the harmonic oscillator subject to a weak damping force can be calculated in the same way as before. We anticipate, though, that the trajectories will not be closed. The motion does not repeat itself, because energy is constantly being dissipated. For the sake of illustration, we assume that the oscillator is started from rest at position $x_0$. The solutions for $x$ and $\dot{x}$ are given by Equations 3.4.21a and b:

\[
\begin{align*}
    x &= A e^{-\gamma t} \sin(\omega_d t + \phi_0) \quad (3.5.4a) \\
    \dot{x} &= -A e^{-\gamma t} (\gamma \sin(\omega_d t + \phi_0) - \omega_d \cos(\omega_d t + \phi_0)) \quad (3.5.4b)
\end{align*}
\]

Remember that because the initial phase angle $\phi_0$ is given by the condition that $\dot{x}_0 = 0$, its value for the damped oscillator is not $\pi/2$ but $\phi_0 = \tan^{-1}\omega_d/\gamma$. It is difficult to eliminate $t$ by brute force in the above parametric equations. Instead, we can illuminate the motion in phase space by applying a sequence of substitutions and linear transformations of the phase-space coordinates that simplifies the above expressions, leading to the form we've already discussed for the harmonic oscillator. First, substitute $\rho = A e^{-\gamma t}$ and

\[
\theta = \omega_d t + \phi_0
\]

into the above equations, obtaining

\[
\begin{align*}
    x &= \rho \sin \theta \quad (3.5.4c) \\
    \dot{x} &= -\rho(\gamma \sin \theta - \omega_d \cos \theta) \quad (3.5.4d)
\end{align*}
\]

Next, we apply the linear transformation $y = \dot{x} + \gamma x$ to Equation 3.5.4d, obtaining

\[
y = \omega_d \rho \cos \theta \quad (3.5.5)
\]
We then square this equation and carry out some algebra to obtain

\[
y^2 = \omega_0^2 \rho^2 (1 - \sin^2 \theta) \\
y^2 = \omega_0^2 (\rho^2 - x^2)
\]  
(3.5.6)

\[
\frac{x^2}{\rho^2} + \frac{y^2}{\omega_0^2 \rho^2} = 1
\]

Voila! Equation 3.5.6 is identical in form to Equation 3.5.2. But here the variable \( y \) is a linear combination of \( x \) and \( \dot{x} \) so the ensemble of points \( (x, y) \) represents a modified phase space. The trajectory of the oscillator in this space is an ellipse whose major and minor axes, characterized by \( \rho \) and \( \omega_0 \rho \), decrease exponentially with time. The trajectory starts off with a maximum value of \( x_0(= A \sin \phi) \) and then spirals inward toward the origin. The result is shown in Figure 3.5.3(a). The behavior of the trajectory in the \( x-\dot{x} \) plane is similar and is shown in Figure 3.5.3(b). Two trajectories are shown in the plots for the cases of strong and weak damping. Which is which should be obvious.

As before, Equation 3.5.6 is none other than the energy equation for the damped harmonic oscillator. We can compare it to the results we obtained in our discussion in Section 3.4 for the rate of energy dissipation in the weakly damped oscillator. In the case of weak damping, the damping factor \( \gamma \) is small compared to \( \omega_0 \) the undamped oscillator angular frequency (see Equation 3.4.10), and, thus, we have

\[
\omega_d = \omega_0 \quad \gamma = \dot{x}
\]  
(3.5.7)

Hence, Equation 3.5.6 becomes

\[
\frac{x^2}{\rho^2} + \frac{\dot{x}^2}{\omega_0^2 \rho^2} = 1
\]  
(3.5.8)

Note that this equation is identical in form to Equation 3.5.6, and consequently the trajectory seen in the \( x-\dot{x} \) plane of Figure 3.5.3(b) for the case of weak damping is virtually identical to the modified phase-space trajectory of the weakly damped oscillator shown in Figure 3.5.3(a). Finally, upon substituting \( k/m \) for \( \omega_0^2 \) and \( A^2 e^{-2\gamma t} \) for \( \dot{\rho}^2 \), we obtain

\[
\frac{1}{2} kx^2 + \frac{1}{2} m\dot{x}^2 = \frac{1}{2} kA^2 e^{-2\gamma t} \\
= \frac{1}{2} m\omega_0^2 A^2 e^{-2\gamma t}
\]  
(3.5.9)

If we compare this result with Equation 3.4.23, we see that it represents the total energy remaining in the oscillator at any subsequent time \( t \):

\[
V(t) + T(t) = E(t)
\]  
(3.5.10)

The energy of the weakly damped harmonic oscillator dies away exponentially with a time constant \( \tau = (2\gamma)^{-1} \). The spiral nature of its phase-space trajectory reflects this fact.

The Critically Damped Harmonic Oscillator

Equation 3.4.9 gave the solution for the critically damped oscillator:

\[
x = (At + B)e^{-\gamma t}
\]  
(3.5.11)
Figure 3.5.3 (a) Modified phase-space plot (see text) for the simple harmonic oscillator. (b) Phase-space plot \( (\omega_0 = 0.5 \text{ s}^{-1}) \). Underdamped case: (1) weak damping \( (\gamma = 0.05 \text{ s}^{-1}) \) and (2) strong damping \( (\gamma = 0.25 \text{ s}^{-1}) \).

Taking the derivative of this equation, we obtain

\[
\dot{x} = -\gamma (\Delta t + B)e^{-\gamma t} + Ae^{-\gamma t}
\]  

(3.5.12)

or

\[
\dot{x} + \gamma x = Ae^{-\gamma t}
\]  

(3.5.13)

This last equation indicates that the phase-space trajectory should approach a straight line whose intercept is zero and whose slope is equal to \(-\gamma\). The phase-space plot is shown in Figure 3.5.4 for motion starting off with the conditions \((x_0, \dot{x}_0) = (1, 0)\).
3.5 Phase Space

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**The Overdamped Oscillator**

Overdamping occurs when the damping parameter $\gamma$ is larger than the angular frequency $\omega_0$. Equation 3.4.6 then gives the solution for the motion:

$$x(t) = A_1 e^{-(\gamma-q)t} + A_2 e^{-(\gamma+q)t}$$  \hspace{1cm} (3.5.14)

in which all the exponents are real. Taking the derivative of this equation, we find

$$\dot{x}(t) = -\gamma x + q e^{-\gamma t}(A_1 e^{\gamma t} - A_2 e^{-\gamma t})$$  \hspace{1cm} (3.5.15)

As in the case of critical damping, the phase path approaches zero along a straight line. However, approaches along two different lines are possible. To see what they are, it is convenient to let the motion start from rest at some displacement $x_0$. Given these conditions, a little algebra yields the following values for $A_1$ and $A_2$:

$$A_1 = \frac{(\gamma + q)}{2q} x_0 \quad A_2 = \frac{-(\gamma - q)}{2q} x_0$$  \hspace{1cm} (3.5.16)

Some more algebra yields the following for two different linear combinations of $x$ and $\dot{x}$:

$$\dot{x} + (\gamma - q)x = (\gamma - q)x_0 e^{-(\gamma - q)t}$$  \hspace{1cm} (3.5.17a)

$$\dot{x} + (\gamma + q)x = (\gamma + q)x_0 e^{-(\gamma + q)t}$$  \hspace{1cm} (3.5.17b)

The term on the right-hand side of each of the above equations dies out with time, and, thus, the phase-space asymptotes are given by the pairs of straight lines:

$$\dot{x} = -(\gamma - q)x$$  \hspace{1cm} (3.5.18a)

$$\dot{x} = -(\gamma + q)x$$  \hspace{1cm} (3.5.18b)

Except for special cases, phase-space paths of the motion always approach zero along the asymptote whose slope is $-(\gamma - q)$. That asymptote invariably "springs into existence" much faster than the other, because its exponential decay factor is $(\gamma + q)$ (Equations 3.5.17), the larger of the two.

Figure 3.5.5 shows the phase-space plot for an overdamped oscillator whose motion starts off with the values $(x_0, \dot{x}_0) = (1, 0)$, along with the asymptote whose slope is $-(\gamma - q)$. Note how rapidly the trajectory locks in on the asymptote, unlike the case of critical damping, where it reaches the asymptote only toward the end of its motion. Obviously, overdamping is the most efficient way to knock the oscillation out of oscillatory motion!
A particle of unit mass is subject to a damping force $-\dot{x}$ and a force that depends on its displacement $x$ from the origin that varies as $+x - x^3$. (a) Find the points of equilibrium of the particle and specify whether or not they are stable or unstable. (b) Use Mathcad to plot phase-space trajectories for the particle for three sets of starting conditions: $(x, y) = (i) (-1, 1.40)$ (ii) $(-1, 1.45)$ (iii) $(0.01, 0)$ and describe the resulting motion.

**Solution:**

(a) The equation of motion is

$$\ddot{x} + x - x^3 = 0$$

Let $y = \dot{x}$. Then

$$\dot{y} = -y + x - x^3$$

At equilibrium, both $y = 0$ and $\dot{y} = 0$. This is satisfied if

$$x - x^3 = x(1 - x^2) = x(1 - x)(1 + x) = 0$$

Thus, there are three equilibrium points $x = 0$ and $x = \pm 1$.

We can determine whether or not they are stable by linearizing the equation of motion for small excursions away from those points. Let $u$ represent a small excursion of the particle away from an equilibrium point, which we designated by $x_0$. Thus, $x = x_0 + u$ and the equation of motion becomes

$$y = \dot{u} \quad \text{and} \quad \dot{y} = -y + (x_0 + u) - (x_0 + u)^3$$

Carrying out the expansion and dropping all terms non-linear in $u$, we get

$$\dot{y} = -y + (1 - 3x_0^2)u + x_0(1 - x_0^2)$$

The last term is zero, so

$$\dot{y} = -y + (1 - 3x_0^2)u$$
If \((1 - 3x^2) < 0\) the motion is a stable, damped oscillation that eventually ceases at \(x = x_0\). If \((1 - 3x^2) > 0\) the particle moves away from \(x_0\) and the equilibrium is unstable. Thus, \(x = \pm 1\) are points of stable equilibrium and \(x_0\) is an unstable point.

(b) The three graphs in Figure 3.5.6 were generated by using Mathcad's \textit{rkfixed} equation solver to solve the complete nonlinear equation of motion numerically. In all cases, no matter how the motion is started, the particle veers away from \(x = 0\) and ultimately terminates at \(x = \pm 1\). The motion for the third set of starting conditions is particularly illuminating. The particle is started at rest near, but not precisely at, \(x = 0\). The particle is repelled away from that point, goes into damped oscillation about \(x = 1\), and eventually comes to rest there. The points \(x = \pm 1\) are called attractors and the point \(x = 0\) is called a repellor.

### 3.6 Forced Harmonic Motion: Resonance

In this section, we study the motion of a damped harmonic oscillator that is subjected to a periodic driving force by an external agent. Suppose a force of the form \(F_0 \cos \omega t\) is exerted upon such an oscillator. The equation of motion is

\[
m \ddot{x} = -kx - c \dot{x} + F_0 \cos \omega t
\]

The most striking feature of such an oscillator is the way in which it responds as a function of the driving frequency even when the driving force is of fixed amplitude. A remarkable phenomenon occurs when the driving frequency is close in value to the natural frequency \(\omega_0\) of the oscillator. It is called resonance. Anyone who has ever pushed a child on a swing knows that the amplitude of oscillation can be made quite large if even the smallest push is made at just the right time. Small, periodic forces exerted on oscillators at frequencies well above or below the natural frequency are much less effective; the amplitude remains small. We initiate our discussion of forced harmonic motion with a qualitative description of the behavior that we might expect. Then we carry out a detailed analysis of the equation of motion (Equation 3.6.1), with our eyes peeled for the appearance of the phenomenon of resonance.

We already know that the undamped harmonic oscillator, subjected to any sort of disturbance that displaces it from its equilibrium position, oscillates at its natural frequency, \(\omega_0 = \sqrt{k/m}\). The dissipative forces inevitably present in any real system changes the frequency of the oscillator slightly, from \(\omega_0\) to \(\omega_d\), and cause the free oscillation to die out. This motion is represented by a solution to the homogeneous differential equation (Equation 3.4.1, which is Equation 3.6.1 without the driving force present). A periodic