4.4 The Harmonic Oscillator in Two and Three Dimensions

Consider the motion of a particle subject to a linear restoring force that is always directed toward a fixed point, the origin of our coordinate system. Such a force can be represented by the expression

$$\mathbf{F} = -k\mathbf{r} \tag{4.4.1}$$

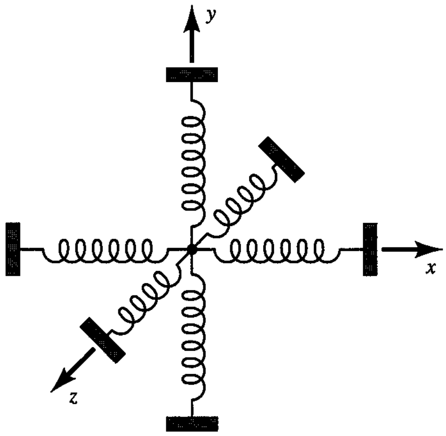

Figure 4.4.1 A model of a three-dimensional harmonic oscillator.

Accordingly, the differential equation of motion is simply expressed as

$$m\frac{d^2\mathbf{r}}{dt^2} = -k\mathbf{r} \tag{4.4.2}$$

The situation can be represented approximately by a particle attached to a set of elastic springs as shown in Figure 4.4.1. This is the three-dimensional generalization of the linear oscillator studied earlier. Equation 4.4.2 is the differential equation of the *linear isotropic oscillator.*

The Two-Dimensional Isotropic Oscillator

In the case of motion in a single plane, Equation 4.4.2 is equivalent to the two component equations

$$\begin{aligned} m\ddot{x} &= -kx \\ m\ddot{y} &= -ky \end{aligned} \tag{4.4.3}$$

These are separated, and we can immediately write down the solutions in the form

$$x = A\cos(\omega t + \alpha) \qquad y = B\cos(\omega t + \beta) \tag{4.4.4}$$

in which

$$\omega = \left(\frac{k}{m}\right)^{1/2} \tag{4.4.5}$$

The constants of integration A, B, α, and β are determined from the initial conditions in any given case.

To find the equation of the path, we eliminate the time t between the two equations. To do this, let us write the second equation in the form

$$y = B\cos(\omega t + \alpha + \Delta) \tag{4.4.6}$$

where

$$\Delta = \beta - \alpha \tag{4.4.7}$$

Then

$$y = B[\cos(\omega t + \alpha) \cos\Delta - \sin(\omega t + \alpha) \sin\Delta] \tag{4.4.8}$$

Combining the above with the first of Equations 4.4.4, we then have

$$\frac{y}{B} = \frac{x}{A} \cos\Delta - \left(1 - \frac{x^2}{A^2}\right)^{1/2} \sin\Delta \tag{4.4.9}$$

and upon transposing and squaring terms, we obtain

$$\frac{x^2}{A^2} - xy\frac{2\cos\Delta}{AB} + \frac{y^2}{B^2} = \sin^2\Delta \tag{4.4.10}$$

which is a quadratic equation in x and y. Now the general quadratic

$$ax^2 + bxy + cy^2 + dx + ey = f \tag{4.4.11}$$

represents an ellipse, a parabola, or a hyperbola, depending on whether the discriminant

$$b^2 - 4ac \tag{4.4.12}$$

is negative, zero, or positive, respectively. In our case the discriminant is equal to $-(2 \sin \Delta/AB)^2$, which is negative, so the path is an ellipse as shown in Figure 4.4.2.

In particular, if the phase difference Δ is equal to $\pi/2$, then the equation of the path reduces to the equation

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1 \tag{4.4.13}$$

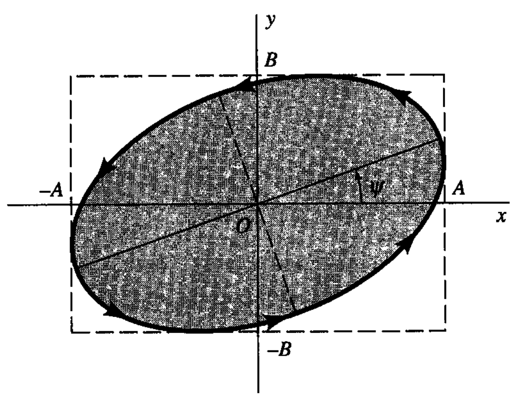

Figure 4.4.2 The elliptical path of a two-dimensional isotropic oscillator.

which is the equation of an ellipse whose axes coincide with the coordinate axes. On the other hand, if the phase difference is 0 or π, then the equation of the path reduces to that of a straight line, namely,

$$y = \pm \frac{B}{A} x \tag{4.4.14}$$

The positive sign is taken if $\Delta = 0$, and the negative sign, if $\Delta = \pi$. In the general case it is possible to show that the axis of the elliptical path is inclined to the x-axis by the angle ψ, where

$$\tan 2\psi = \frac{2AB\cos\Delta}{A^2 - B^2} \tag{4.4.15}$$

The derivation is left as an exercise.

The Three-Dimensional Isotropic Harmonic Oscillator

In the case of three-dimensional motion, the differential equation of motion is equivalent to the three equations

$$m\ddot{x} = -kx \qquad m\ddot{y} = -ky \qquad m\ddot{z} = -kz \tag{4.4.16}$$

which are separated. Hence, the solutions may be written in the form of Equations 4.4.4, or, alternatively, we may write

$$\begin{aligned} x &= A_1 \sin\omega t + B_1 \cos\omega t \\ y &= A_2 \sin\omega t + B_2 \cos\omega t \\ z &= A_3 \sin\omega t + B_3 \cos\omega t \end{aligned} \tag{4.4.17a}$$

The six constants of integration are determined from the initial position and velocity of the particle. Now Equations 4.4.16 can be expressed vectorially as

$$\mathbf{r} = \mathbf{A}\sin\omega t + \mathbf{B}\cos\omega t \tag{4.4.17b}$$

in which the components of $\mathbf{A}$ are A_1, A_2, and A_3, and similarly for $\mathbf{B}$. It is clear that the motion takes place entirely in a single plane, which is common to the two constant vectors $\mathbf{A}$ and $\mathbf{B}$, and that the path of the particle in that plane is an ellipse, as in the two-dimensional case. Hence, the analysis concerning the shape of the elliptical path under the two-dimensional case also applies to the three-dimensional case.

Nonisotropic Oscillator

The previous discussion considered the motion of the isotropic oscillator, wherein the restoring force is independent of the direction of the displacement. If the magnitudes of the components of the restoring force depend on the direction of the displacement, we have the case of the *nonisotropic oscillator*. For a suitable choice of axes, the differential equations for the nonisotropic case can be written

$$\begin{aligned} m\ddot{x} &= -k_1 x \\ m\ddot{y} &= -k_2 y \\ m\ddot{z} &= -k_3 z \end{aligned} \tag{4.4.18}$$

Here we have a case of *three* different frequencies of oscillation, $\omega_1 = \sqrt{k_1/m}$, $\omega_2 = \sqrt{k_2/m}$, and $\omega_3 = \sqrt{k_3/m}$, and the motion is given by the solutions

$$\begin{aligned} x &= A\cos(\omega_1 t + \alpha) \\ y &= B\cos(\omega_2 t + \beta) \\ z &= C\cos(\omega_3 t + \gamma) \end{aligned} \tag{4.4.19}$$

Again, the six constants of integration in the above equations are determined from the initial conditions. The resulting oscillation of the particle lies entirely within a rectangular box (whose sides are $2A$, $2B$, and $2C$) centered on the origin. In the event that ω_1, ω_2, and ω_3 are commensurate—that is, if

$$\frac{\omega_1}{n_1} = \frac{\omega_2}{n_2} = \frac{\omega_3}{n_3} \tag{4.4.20}$$

where n_1, n_2, and n_3 are integers—the path, called a *Lissajous* figure, is closed, because after a time $2\pi n_1/\omega_1 = 2\pi n_2/\omega_2 = 2\pi n_3/\omega_3$ the particle returns to its initial position and the motion is repeated. (In Equation 4.4.20 we assume that any common integral factor is canceled out.) On the other hand, if the ω's are *not* commensurate, the path is not closed. In this case the path may be said to completely fill the rectangular box mentioned above, at least in the sense that if we wait long enough, the particle comes arbitrarily close to any given point.

The net restoring force exerted on a given atom in a solid crystalline substance is approximately linear in the displacement in many cases. The resulting frequencies of oscillation usually lie in the infrared region of the spectrum: 10^{12} to 10^{14} vibrations per second.

Energy Considerations

In the preceding chapter we showed that the potential energy function of the one-dimensional harmonic oscillator is quadratic in the displacement, $V(x) = \frac{1}{2}kx^2$. For the general three-dimensional case, it is easy to verify that

$$V(x,y,z) = \tfrac{1}{2}k_1x^2 + \tfrac{1}{2}k_2y^2 + \tfrac{1}{2}k_3z^2 \tag{4.4.21}$$

because $F_x = -\partial V/\partial x = -k_1 x$, and similarly for F_y and F_z. If $k_1 = k_2 = k_3 = k$, we have the isotropic case, and

$$V(x,y,z) = \tfrac{1}{2}k(x^2 + y^2 + z^2) = \tfrac{1}{2}kr^2 \tag{4.4.22}$$

The total energy in the isotropic case is then given by the simple expression

$$\frac{1}{2}mv^2 + \frac{1}{2}kr^2 = E \tag{4.4.23}$$

which is similar to that of the one-dimensional case discussed in the previous chapter.

EXAMPLE 4.4.1

A particle of mass m moves in two dimensions under the following potential energy function:

$$V(\mathbf{r}) = \frac{1}{2}k(x^2 + 4y^2)$$

Find the resulting motion, given the initial condition at $t = 0$: $x = a$, $y = 0$, $\dot{x} = 0$, $\dot{y} = v_0$.

Solution:

This is a nonisotropic oscillator potential. The force function is

$$\mathbf{F} = -\nabla V = -\mathbf{i}kx - \mathbf{j}4ky = m\ddot{\mathbf{r}}$$

The component differential equations of motion are then

$$m\ddot{x} + kx = 0 \qquad m\ddot{y} + 4ky = 0$$

The x-motion has angular frequency $\omega = (k/m)^{1/2}$, while the y-motion has angular frequency just twice that, namely, $\omega_y = (4k/m)^{1/2} = 2\omega$. We shall write the general solution in the form

$$x = A_1 \cos\omega t + B_1 \sin\omega t$$
$$y = A_2 \cos 2\omega t + B_2 \sin 2\omega$$

To use the initial condition we must first differentiate with respect to t to find the general expression for the velocity components:

$$\dot{x} = -A_1\omega \sin\omega t + B_1\omega \cos\omega t$$
$$\dot{y} = -2A_2\omega \sin 2\omega t + 2B_2\omega \cos 2\omega t$$

Thus, at $t = 0$, we see that the above equations for the components of position and velocity reduce to

$$a = A_1 \qquad 0 = A_2 \qquad 0 = B_1\omega \qquad v_0 = 2B_2\omega$$

These equations give directly the values of the amplitude coefficients, $A_1 = a$, $A_2 = B_1 = 0$, and $B_2 = v_0/2\omega$, so the final equations for the motion are

$$x = a \cos\omega t$$
$$y = \frac{v_0}{2\omega} \sin 2\omega t$$

The path is a Lissajous figure having the shape of a figure-eight as shown in Figure 4.4.3.

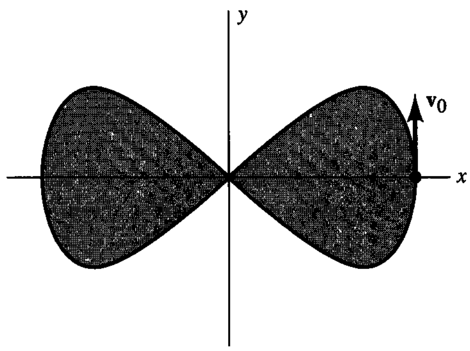

Figure 4.4.3 A Lissajous figure.

4.5 | Motion of Charged Particles in Electric and Magnetic Fields

When an electrically charged particle is in the vicinity of other electric charges, it experiences a force. This force $\mathbf{F}$ is said to be caused by the electric field $\mathbf{E}$, which arises from these other charges. We write

$$\mathbf{F} = q\mathbf{E} \tag{4.5.1}$$

where q is the electric charge carried by the particle in question.[6] The equation of motion of the particle is then

$$m\frac{d^2\mathbf{r}}{dt^2} = q\mathbf{E} \tag{4.5.2a}$$

or, in component form,

$$\begin{aligned} m\ddot{x} &= qE_x \\ m\ddot{y} &= qE_y \\ m\ddot{z} &= qE_z \end{aligned} \tag{4.5.2b}$$

The field components are, in general, functions of the position coordinates x, y, and z. In the case of time-varying fields (that is, if the charges producing $\mathbf{E}$ are moving), the components also involve t.

Let us consider a simple case, namely, that of a uniform constant electric field. We can choose one of the axes—say, the z-axis—to be in the direction of the field. Then $E_x = E_y = 0$, and $E = E_z$. The differential equations of motion of a particle of charge q moving in this field are then

$$\ddot{x} = 0 \qquad \ddot{y} = 0 \qquad \ddot{z} = \frac{qE}{m} = \text{constant} \tag{4.5.3}$$

[6] In SI units, F is in newtons, q in coulombs, and E in volts per meter.

These are of exactly the same form as those for a projectile in a uniform gravitational field. The path is, therefore, a parabola, if $\dot{x}$ and $\dot{y}$ are not both zero initially. Otherwise, the path is a straight line, as with a body falling vertically.

Textbooks dealing with electromagnetic theory[7] show that

$$\nabla \times \mathbf{E} = 0 \tag{4.5.4}$$

if $\mathbf{E}$ is due to static charges. This means that motion in such a field is conservative, and that there exists a potential function Φ such that $\mathbf{E} = -\nabla\Phi$. The potential energy of a particle of charge q in such a field is then $q\Phi$, and the total energy is constant and is equal to $\frac{1}{2}mv^2 + q\Phi$.

In the presence of a static magnetic field $\mathbf{B}$ (called the magnetic induction), the force acting on a moving particle is conveniently expressed by means of the cross product, namely,

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B}) \tag{4.5.5}$$

where $\mathbf{v}$ is the velocity and q is the charge.[8] The differential equation of motion of a particle moving in a purely magnetic field is then

$$m\frac{d^2\mathbf{r}}{dt^2} = q(\mathbf{v} \times \mathbf{B}) \tag{4.5.6}$$

Equation 4.5.6 states that the acceleration of the particle is always at right angles to the direction of motion. This means that the tangential component of the acceleration $(\dot{v})$ is zero, and so the particle moves with constant speed. This is true even if $\mathbf{B}$ is a varying function of the position $\mathbf{r}$, as long as it does not vary with time.

EXAMPLE 4.5.1

Let us examine the motion of a charged particle in a uniform constant magnetic field. Suppose we choose the z-axis to be in the direction of the field; that is, we write

$$\mathbf{B} = \mathbf{k}B$$

The differential equation of motion now reads

$$m\frac{d^2\mathbf{r}}{dt^2} = q(\mathbf{v} \times \mathbf{k}B) = qB\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & 0 & 1 \end{vmatrix}$$

$$m(\mathbf{i}\ddot{x} + \mathbf{j}\ddot{y} + \mathbf{k}\ddot{z}) = qB(\mathbf{i}\dot{y} - \mathbf{j}\dot{x})$$

[7] For example, Reitz, Milford, and Christy, op cit.

[8] Equation 4.5.5 is valid for SI units: F is in newtons, q in coulombs, v in meters per second, and B in webers per square meter.

Equating components, we have

$$\begin{aligned} m\ddot{x} &= qB\dot{y} \\ m\ddot{y} &= -qB\dot{x} \\ \ddot{z} &= 0 \end{aligned} \tag{4.5.7}$$

Here, for the first time we meet a set of differential equations of motion that are *not* of the separated type. The solution is relatively simple, however, for we can integrate at once with respect to t, to obtain

$$\begin{aligned} m\dot{x} &= qBy + c_1 \\ m\dot{y} &= -qBx + c_2 \\ \dot{z} &= \text{constant} = \dot{z}_0 \end{aligned}$$

or

$$\dot{x} = \omega y + C_1 \qquad \dot{y} = -\omega x + C_2 \qquad \dot{z} = \dot{z}_0 \tag{4.5.8}$$

where we have used the abbreviation $\omega = qB/m$. The c's are constants of integration, and $C_1 = c_1/m$, $C_2 = c_2/m$. Upon inserting the expression for $\dot{y}$ from the second part of Equation 4.5.8 into the first part of Equation 4.5.7, we obtain the following separated equation for x:

$$\ddot{x} + \omega^2 x = \omega^2 a \tag{4.5.9}$$

where $a = C_2/\omega$. The solution is

$$x = a + A\cos(\omega t + \theta_0) \tag{4.5.10}$$

where A and θ_0 are constants of integration. Now, if we differentiate with respect to t, we have

$$\dot{x} = -A\omega\sin(\omega t + \theta_0) \tag{4.5.11}$$

The above expression for $\dot{x}$ may be substituted for the left-hand side of the first of Equations 4.5.8 and the resulting equation solved for y. The result is

$$y = b - A\sin(\omega t + \theta_0) \tag{4.5.12}$$

where $b = -C_1/\omega$. To find the form of the path of motion, we eliminate t between Equation 4.5.10 and Equation 4.5.12 to get

$$(x-a)^2 + (y-b)^2 = A^2 \tag{4.5.13}$$

Thus, the projection of the path of motion on the xy plane is a circle of radius A centered at the point (a, b). Because, from the third of Equations 4.5.8, the speed in the z direction is constant, we conclude that the path is a *helix*. The axis of the winding path is in the direction of the magnetic field, as shown in Figure 4.5.1. From Equation 4.5.12 we have

$$\dot{y} = -A\omega\cos(\omega t + \theta_0) \tag{4.5.14}$$

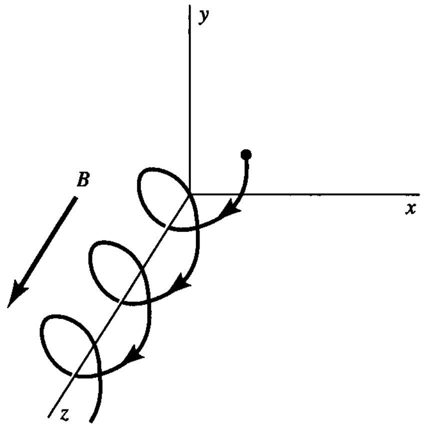

Figure 4.5.1 The helical path of a particle moving in a magnetic field.

Upon eliminating t between Equation 4.5.11 and Equation 4.5.14, we find

$$\dot{x}^2 + \dot{y}^2 = A^2\omega^2 = A^2\left(\frac{qB}{m}\right)^2 \tag{4.5.15}$$

Letting $v_1 = (\dot{x}^2 + \dot{y}^2)^{1/2}$, we see that the radius A of the helix is given by

$$A = \frac{v_1}{\omega} = v_1 \frac{m}{qB} \tag{4.5.16}$$

If there is no component of the velocity in the z direction, the path is a circle of radius A. It is evident that A is directly proportional to the speed v_1 and that the angular frequency ω of motion in the circular path is independent of the speed. The angular frequency ω is known as the cyclotron frequency. The cyclotron, invented by Ernest Lawrence, depends for its operation on the fact that ω is independent of the speed of the charged particle.

4.6 Constrained Motion of a Particle

When a moving particle is restricted geometrically in the sense that it must stay on a certain definite surface or curve, the motion is said to be *constrained.* A piece of ice sliding around a bowl and a bead sliding on a wire are examples of constrained motion. The constraint may be complete, as with the bead, or it may be one-sided, as with the ice in the bowl. Constraints may be fixed, or they may be moving. In this chapter we study only fixed constraints.

The Energy Equation for Smooth Constraints

The total force acting on a particle moving under constraint can be expressed as the vector sum of the net external force $\mathbf{F}$ and the force of constraint $\mathbf{R}$. The latter force is the reaction of the constraining agent upon the particle. The equation of motion may, therefore, be written

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} + \mathbf{R} \tag{4.6.1}$$