CHAPTER 12

Coupled Oscillations

In this Chapter, we will explore the properties of c*oupled harmonic oscillators*. These systems can be analyzed by using either the Lagrangian approach of Chapter 8, or alternatively using Newton's Second Law. The simplest form of these systems in mechanics contains two masses connected by springs to each other. A second simple example of coupled mechanical oscillators is the double pendulum, which also exhibits a wide range of interesting behaviors. We will see that these simple oscillating systems can exhibit normal modes of oscillation, which are patterns of motion in which all parts of the system move sinusoidally with the same frequency. The frequencies of the normal modes of a system are known as its natural frequencies of oscillation. We will find that any motion exhibited by the system can be expressed as a linear combination of these normal modes.

The discussion of the two-mass system will lead us to a more general description of linearly coupled harmonic systems, and how their equations of motion can be written in matrix form. The best way to obtain solutions to the equations of motions for coupled oscillations is by using standard techniques from Linear Algebra, in order to find the eigenvalues and eigenvectors of a matrix. The eigenvectors and eigenvalues of the matrix characterizing the oscillating system are closely related to its normal modes.

This Chapter will conclude with a general treatment of coupled oscillations, and a discussion of normal coordinates.

12.1 COUPLED OSCILLATIONS OF A TWO-MASS THREE-SPRING SYSTEM

12.1.1 The Equations of Motion - Numerical Solution

Consider two masses m_1 and m_2 attached to three springs with spring constants k_1, k_2, and k_3 and to the two fixed walls, as shown in Figure 12.1.

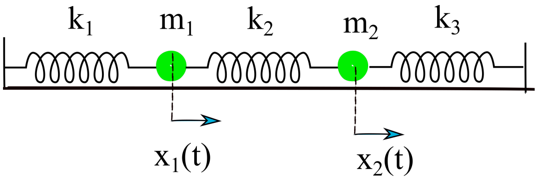


Figure 12.1: A system of two coupled harmonic oscillators consisting of two masses m_1 and m_2 connected with three springs with constants k_1, k_2, and k_3.

Let us denote by $x_1(t)$ and $x_2(t)$ the horizontal displacements of the two masses from their respective equilibrium points. The force on the first mass due to the first spring is $-k_1x_1$. The middle spring will be stretched by a distance $(x_1 - x_2)$, and the force on the first mass due to this middle spring will be $-k_2(x_1 - x_2)$. The total force on the first mass must then be $F_1 = -k_1x_1 - k_2(x_1 - x_2)$. Similarly, the force on the second mass due to the middle spring is $-k_2(x_2 - x_1)$, and the force on the second mass due to this third spring will be $-k_3x_2$.

Using the notation $\ddot{x}$ for acceleration, the equations of motion from Newton's Second Law $F = ma$ for the two masses are:

$$\left.\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 - k_2(x_1 - x_2) \\ m_2\ddot{x}_2 &= -k_2(x_2 - x_1) - k_3x_2 \end{aligned}\right\} \tag{12.1.1}$$

In a more general way, we can obtain the same equations by starting with the Lagrangian formulation of Chapter 8. The potential energies of the two end springs are $V_1 = k_1x_1^2/2$ and $V_3 = k_3x_2^2/2$, while the potential energy for the middle spring is $V_2 = k_2(x_1 - x_2)^2/2$, so that the Lagrangian is equal to:

$$\mathcal{L} = T - V_{Total} = \frac{1}{2}m_1\dot{x_1}^2 + \frac{1}{2}m_2\dot{x_2}^2 - \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_3x_2^2 - \frac{1}{2}k_2(x_1 - x_2)^2 \tag{12.1.2}$$

The Euler-Lagrange equations are:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right) - \frac{\partial \mathcal{L}}{\partial x_1} = 0 \quad \rightarrow \quad m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2) \tag{12.1.3}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2}\right) - \frac{\partial \mathcal{L}}{\partial x_2} = 0 \quad \rightarrow \quad m_2\ddot{x}_2 = -k_2(x_2 - x_1) - k_3x_2 \tag{12.1.4}$$

These are of course the same equations as in (12.1.1).

In general, it is not possible to obtain the solutions $x_1(t)$ and $x_2(t)$ of the system of equations (12.1.1) analytically, and they must be obtained by numerically integrating the equations for given initial conditions of the system. The initial conditions are usually given as the initial positions and initial speeds of the two masses.

Example 12.1 shows how to obtain and plot the numerical solutions $x_1(t)$ and $x_2(t)$, with the initial conditions $x_1(0) = 0$, $x_2(0) = a$, $\dot{x}_1(0) = a$, and $\dot{x}_2(0) = 0$.

Example 12.1: Numerical solution for the general case of two coupled oscillating masses

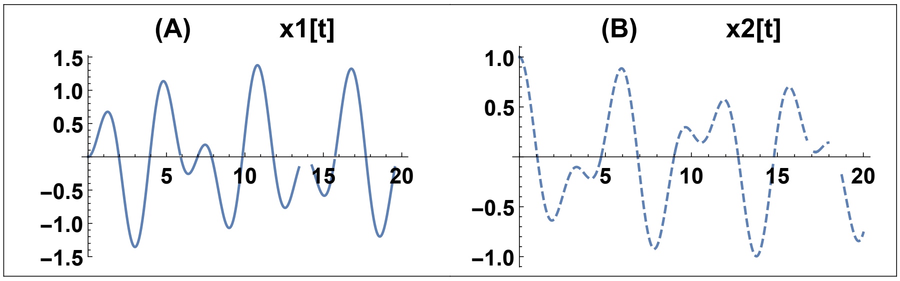
Integrate (12.1.1) for $k_1 = 1\,\text{N/m}$, $k_2 = 2\,\text{N/m}$, $k_3 = 1\,\text{N/m}$, $m_1 = 1\,\text{kg}$, $m_2 = 2\,\text{kg}$ and plot the numerical solutions $x_1(t)$ and $x_2(t)$, with the initial conditions $x_1(0) = 0$, $x_2(0) = 1\,\text{m}$, $\dot{x}_1(0) = 0$ and $\dot{x}_2(0) = 0$. This situation corresponds to the case where the first mass m_1 is initially at rest at its equilibrium position ($x_1(0) = 0$ and $\dot{x}_1(0) = 0$), and the second mass is pulled a distance $a = 1$ meter from its equilibrium and released from rest ($\dot{x}_2(0) = 0$).

Solution:

In this example we can use either the *NDSolve* command in Mathematica to numerically solve the differential equations, or alternatively use the *DSolve* command to obtain the analytical solutions $x_1(t)$ and $x_2(t)$. Although Mathematica can obtain analytical expressions for $x_1(t)$ and $x_2(t)$, they are not listed here because they are algebraically very complex, and therefore the output is suppressed in this example by using the semicolon (;) at the end of the command line.

The parameter *numValues* in the code contains the numerical values for the parameters m_1, m_2, k_1, k_2, and k_3 in the form of a rule ($\rightarrow$). These numerical values are needed in order to plot the solutions using the *Plot* and *GraphicsGrid* commands.

```
sol = DSolve[{m1 * x1"[t] == -k1 * x1[t] + k2 * (x2[t] - x1[t]),
m2 * x2"[t] == -k3 * x2[t] + k2 * (x1[t] - x2[t]), x1[0] == 0, x1'[0] == 0,
x2[0] == a, x2'[0] == 0}, {x1[t], x2[t]}, t];

numValues = {a → 1, k1 → 1, k2 → 2, k3 → 3, m1 → 1, m2 → 2};

gr1 = Plot[x1[t]/.sol/.numValues, {t, 0, 20}, PlotLabel → "(A) x1[t]"];
gr2 = Plot[x2[t]/.sol/.numValues, {t, 0, 20}, PlotStyle → Dashed,
PlotLabel → "(B) x2[t]"];

GraphicsGrid[{{gr1, gr2}}, Frame → True, ImageSize → Large]
```



The plots of $x_1(t)$ and $x_2(t)$ in Example 12.1 are obviously complex, and it is not possible to give a simple physical description of the motion of the two masses. The key physical component which creates this complex behavior is the middle spring in Figure 12.1 since this is the component that couples the motion of the two masses.

12.1.2 Equal Masses and Identical Springs: The Normal Modes

In Examples 12.2 and 12.3, we look at the special case of equal masses ($m_1 = m_2$), and springs with equal spring constants ($k_1 = k_2 = k_3$). This is an interesting physical situation, in which the analytical solutions are simple, and it may be easy to understand the physics of the situation. These two examples introduce the concept of normal modes in a simple and clear manner.

Example 12.2: Equal masses and identical springs: the symmetric oscillation mode

Integrate (12.1.1), and plot the numerical solutions $x_1(t)$ and $x_2(t)$ for the special case of identical masses and identical springs, $m_1 = m_2$ and $k_1 = k_2 = k_3$. Consider the physical situation with the initial conditions $x_1(0) = a$, $x_2(0) = a$, $\dot{x}_1(0) = 0$ and $\dot{x}_2(0) = 0$. Plot the solutions with Mathematica, by using the numerical values $a = 1.0\,\text{m}$, $k = 1.0\,\text{N/m}$, $m = 1.0\,\text{kg}$.

Solution:

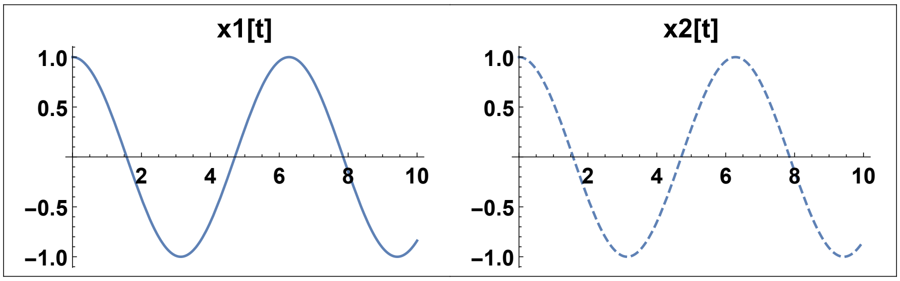
This situation corresponds to the case where the two masses are pulled the same distance a from their corresponding equilibrium, and are released from rest ($\dot{x}_1(0) = 0$

and $\dot{x}_2(0) = 0$). The Mathematica code uses the *DSolve* command to obtain analytical expression for $x_1(t)$ and $x_2(t)$, and the *ExpToTrig* command is used to convert the solutions $x_1(t)$ and $x_2(t)$ into a form containing trigonometric functions instead of exponential functions. The result of *DSolve* appears above the graph.

The analytical solutions are $x_1(t) = x_2(t) = a\cos\left(\sqrt{k/m}\,t\right)$. This tells us that if the two masses are initially displaced from equilibrium by the same distance and released from rest, the two masses will move together with the same speed and in phase as if the middle spring was not present. This makes physical sense, since in this situation the middle spring will be unstretched from its natural length, and will remain unstretched during the motion of the two masses. The frequency of oscillation for both masses in this situation is $\omega_1 = \sqrt{k/m}$, *i.e.* the same frequency as if only one of the two masses were attached to a single spring with a spring constant k.

```
sol =
ExpToTrig[DSolve[{m * x1''[t] == -k * x1[t] + k * (x2[t] - x1[t]), m * x2''[t] ==
-k * x2[t] + k * (x1[t] - x2[t]), x1[0] == a, x1'[0] == 0, x2[0] == a, x2'[0] ==
0}, {x1[t], x2[t]}, t]]//Simplify

numValues = {a → 1, k → 1, m → 1};

gr1 = Plot[x1[t]/.sol/.numValues, {t, 0, 10}, PlotLabel → "x1[t]"];
gr2 = Plot[x2[t]/.sol/.numValues, {t, 0, 10}, PlotLabel → "x2[t]", PlotStyle → Dashed];

GraphicsGrid[{{gr1, gr2}}, Frame → True, ImageSize → Large]

OUTPUT:
```

$\left\{\left\{x1[t] \to a\mathrm{Cos}\left[\frac{\sqrt{k}t}{\sqrt{m}}\right], x2[t] \to a\mathrm{Cos}\left[\frac{\sqrt{k}t}{\sqrt{m}}\right]\right\}\right\}$



Example 12.3: Equal masses and identical springs: the antisymmetric oscillation

Repeat Example 12.2, by using a different set of initial conditions $x_1(0) = a$, $x_2(0) = -a$, $\dot{x}_1(0) = 0$ and $\dot{x}_2(0) = 0$. Plot the solutions with Mathematica by using the numerical values $a = 1.0\,\mathrm{m}$, $k = 1.0\,\mathrm{N/m}$, $m = 1.0\,\mathrm{kg}$.

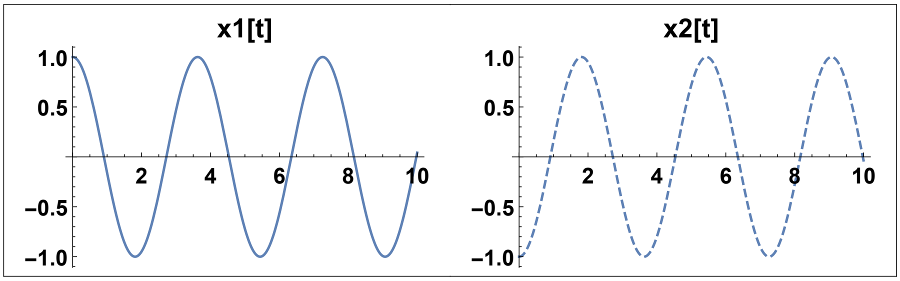
Solution:

In this situation the two masses are initially displaced from their equilibrium positions by equal and *opposite* distances a, and are then released from rest.

The analytical solutions (appearing above the graph) in this case from Mathematica are $x_1(t) = a\cos\left(\sqrt{3k/m}\,t\right)$ and $x_2(t) = -a\cos\left(\sqrt{3k/m}\,t\right)$. This tells us that the two masses will move together with the same speed, but they will be completely out of phase as shown in the output of the code. The frequency of oscillation for both masses in this situation is $\omega_2 = \sqrt{3k/m}$.

```
sol =
ExpToTrig[DSolve[{m * x1''[t] == -k * x1[t] + k * (x2[t] - x1[t]), m * x2''[t] ==
-k * x2[t] + k * (x1[t] - x2[t]), x1[0] == a, x1'[0] == 0, x2[0] == -a, x2'[0] ==
0}, {x1[t], x2[t]}, t]]//Simplify

numValues = {a -> 1, k -> 1, m -> 1};

gr1 = Plot[x1[t]/.sol/.numValues, {t, 0, 10}, PlotLabel -> "x1[t]"];
gr2 = Plot[x2[t]/.sol/.numValues, {t, 0, 10}, PlotLabel -> "x2[t]", PlotStyle -> Dashed];

GraphicsGrid[{{gr1, gr2}}, Frame -> True, ImageSize -> Large]

OUTPUT:
```

$$\left\{\left\{x1[t] \to a\mathrm{Cos}\left[\frac{\sqrt{3}\sqrt{k}t}{\sqrt{m}}\right], x2[t] \to -a\mathrm{Cos}\left[\frac{\sqrt{3}\sqrt{k}t}{\sqrt{m}}\right]\right\}\right\}$$



Examples 12.2 and 12.3 show that the system of two equal masses and three identical springs in Figure 12.1 has two *natural frequencies* given by $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{3k/m}$. By properly choosing the initial conditions in the system as in the two examples, we can force both masses to oscillate with a single frequency, either ω_1 or ω_2. In these special situations, the two natural frequencies are uncoupled from each other, and we say that these are the *normal modes* of the oscillating system.

Figure 12.2 shows schematically the motion of the two masses in either the *symmetric* oscillation pattern with frequency $\omega_1 = \sqrt{k/m}$ (left panel), or an *antisymmetric* oscillation with frequency $\omega_2 = \sqrt{3k/m}$ (right panel).

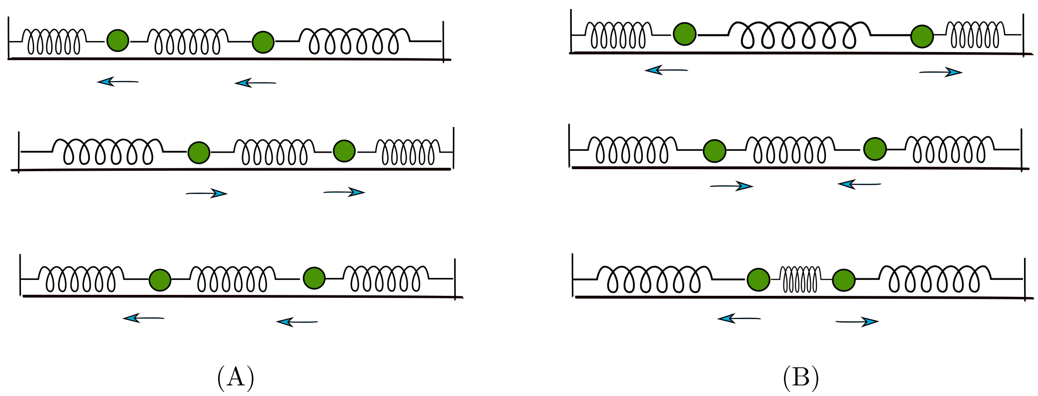


Figure 12.2: The normal modes of the two mass-three spring system in Figure 12.1, corresponding to the natural frequencies $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{3k/m}$. These normal modes represent *symmetric* and *antisymmetric* oscillations respectively.

12.1.3 The General Case: Linear Combination of Normal Modes

If the two masses are displaced at some random distances and are released, they will oscillate in a complex manner, which can be described as the linear combination of oscillations with frequencies ω_1 and ω_2.

The Mathematica code in Example 12.4 shows a general case, where the solutions $x_1(t)$ and $x_2(t)$ are indeed linear combinations of trigonometric functions involving the two frequencies $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{3k/m}$. This example shows how we can numerically decouple the motions corresponding to these two frequencies.

Example 12.4: Analytical solutions for equal masses and identical springs; decoupling of the two frequencies

Integrate (12.1.1) for identical springs and identical masses, with the initial conditions $x_1(0) = 0$, $x_2(0) = a$, $\dot{x}_1(0) = 0$ and $\dot{x}_2(0) = 0$.

Solution:

In this case, the second mass in Figure 12.1 is moved from equilibrium by a distance a and is released from rest. The first mass is initially at rest at its equilibrium position.

The analytical solutions obtained from Mathematica (appearing above the graphs) are

$$x_1(t) = (a/2)\left[\cos\left(\sqrt{k/m}\,t\right) - \cos\left(\sqrt{3k/m}\,t\right)\right]$$

$$x_2(t) = a/2\left[\cos\left(\sqrt{k/m}\,t\right) + \cos\left(\sqrt{3k/m}\,t\right)\right]$$

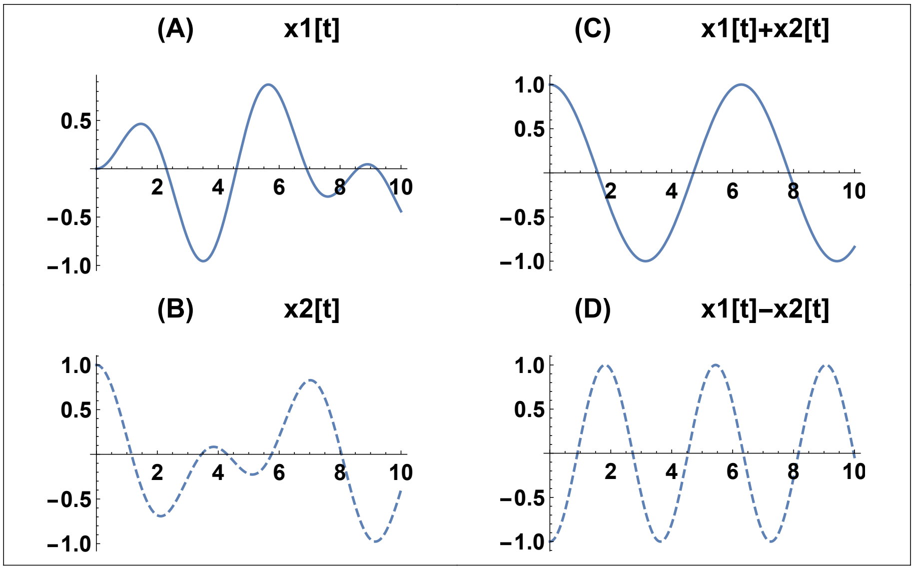
and these are shown in panels (A) and (B) below. Once more the functions $x_1(t)$ and $x_2(t)$ are complicated, and it is difficult to describe how exactly the two masses are moving. This is because mathematically both $x_1(t)$ and $x_2(t)$ contain the frequencies ω_1 and ω_2.

Panels (C) and (D) show plots of the function $x_1(t) + x_2(t) = a\cos\left(\sqrt{k/m}\,t\right)$ and $x_1(t) - x_2(t) = a\cos\left(\sqrt{3k/m}\,t\right)$, respectively. By using these linear combinations, it is now possible to *decouple* the two normal modes, so that the motions shown in panels (C) and (D) are simple cosine functions with frequencies $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{3k/m}$.

```
sol =
ExpToTrig[DSolve[{m * x1"[t] == -k * x1[t] + k * (x2[t] - x1[t]), m * x2"[t] ==
-k * x2[t] + k * (x1[t] - x2[t]), x1[0] == 0, x1'[0] == 0, x2[0] == a, x2'[0] ==
0}, {x1[t], x2[t]}, t]]//Simplify

numValues = {a → 1, k → 1, m → 1};

gr1 = Plot[x1[t]/.sol/.numValues, {t, 0, 10}, BaseStyle → {Bold, FontSize →
14}, PlotLabel → "(A) x1[t]"];
gr2 = Plot[x2[t]/.sol/.numValues, {t, 0, 10}, BaseStyle → {Bold, FontSize →
14}, PlotStyle → Dashed,
PlotLabel → "(B) x2[t]"];
gr3 = Plot[(x1[t] + x2[t])/.sol/.numValues, {t, 0, 10}, BaseStyle → {Bold, FontSize → 14},
PlotLabel → "(C) x1[t]+x2[t]"];
gr4 = Plot[(x1[t] - x2[t])/.sol/.numValues, {t, 0, 10}, BaseStyle → {Bold, FontSize →
14}, PlotStyle → Dashed,
PlotLabel → "(D) x1[t]-x2[t]"];

test = GraphicsGrid[{{gr1, gr3}, {gr2, gr4}}, Frame → True, ImageSize → Large]
```

OUTPUT:

$$\left\{\left\{x1[t] \to \frac{1}{2}a\left(\mathrm{Cos}\left[\frac{\sqrt{k}t}{\sqrt{m}}\right] - \mathrm{Cos}\left[\frac{\sqrt{3}\sqrt{k}t}{\sqrt{m}}\right]\right), x2[t] \to \frac{1}{2}a\left(\mathrm{Cos}\left[\frac{\sqrt{k}t}{\sqrt{m}}\right] + \mathrm{Cos}\left[\frac{\sqrt{3}\sqrt{k}t}{\sqrt{m}}\right]\right)\right\}\right\}$$



In Example 12.5 we examine one more interesting behavior of the two-mass system, the case of *weakly coupled oscillators*. In this example the weak coupling is established by choosing a middle spring with smaller spring constant than the two end springs ($k_1 = k_3 = k$ and $k_2 << k$)

Example 12.5: Weakly coupled oscillators

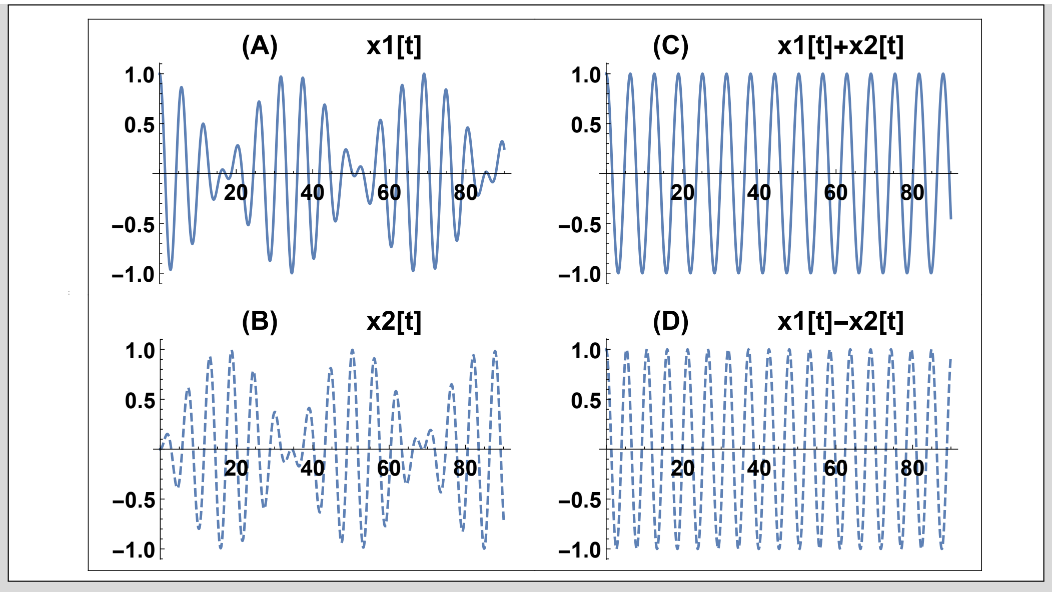
Integrate (12.1.1) and plot the numerical solutions $x_1(t)$ and $x_2(t)$ for the special case of identical masses ($m_1 = m_2 = 1\,\text{kg}$), identical end springs ($k_1 = k_3 = 1\,\text{N/m}$), and a smaller middle spring constant ($k_2 = 0.2\,\text{N/m}$). Consider two physical situations with the initial conditions $x_1(0) = 1\,\text{m}$, $x_2(0) = 0$, $\dot{x}_1(0) = 0$, and $\dot{x}_2(0) = 0$.

Solution:

The Mathematica code below shows the solutions $x_1(t)$ and $x_2(t)$ in panels (A) and (B). The motion of the two masses shows a clear beat pattern, in which mass m_1 reaches a maximum amplitude of oscillation at the same time that m_2 reaches a minimum amplitude, and vice versa.

In this example it is again possible to *uncouple* the motions of the two masses, by plotting the sum $x_1(t)+x_2(t)$ and the difference $x_1(t)-x_2(t)$ of the two functions $x_1(t)$ and $x_2(t)$. This is shown in panels (C) and (D) of the figure, where the two linear combinations $x_1(t)+x_2(t)$ and $x_1(t)-x_2(t)$ can be seen to have pure harmonic oscillations with different frequencies, corresponding to the two normal modes of the system. We will see a more general method of uncoupling the normal modes of the system later in this Chapter, when we discuss the concept of normal coordinates.

```
sol = ExpToTrig[DSolve[{m * x1''[t] == -k1 * x1[t] + k2 * (x2[t] - x1[t]), m * x2''[t] ==
-k3 * x2[t] + k2 * (x1[t] - x2[t]), x1[0] == a, x1'[0] == 0,
x2[0] == 0, x2'[0] == 0}, {x1[t], x2[t]}, t]]//Simplify;

numValues = {a -> 1, k1 -> 1, k2 -> .2, k3 -> 1, m -> 1};

gr1 = Plot[x1[t]/.sol/.numValues, {t, 0, 90}, BaseStyle -> {Bold, FontSize -> 14},
PlotLabel -> "(A) x1[t]"];
gr2 = Plot[x2[t]/.sol/.numValues, {t, 0, 90}, BaseStyle -> {Bold, FontSize -> 14},
PlotStyle -> Dashed, PlotLabel -> "(B) x2[t]"];
gr3 = Plot[(x1[t] + x2[t])/.sol/.numValues, {t, 0, 90},
BaseStyle -> {Bold, FontSize -> 14}, PlotLabel -> "(C) x1[t]+x2[t]"];
gr4 = Plot[(x1[t] - x2[t])/.sol/.numValues, {t, 0, 90},
BaseStyle -> {Bold, FontSize -> 14}, PlotStyle -> Dashed,
PlotLabel -> "(D) x1[t]-x2[t]"];

GraphicsGrid[{{gr1, gr3}, {gr2, gr4}}, Frame -> True, ImageSize -> Large]
```



In the next two sections, we develop a more formal mathematical analysis of the normal modes for the system in Figure 12.1 by using the techniques of Linear Algebra.

12.2 NORMAL MODE ANALYSIS OF THE TWO-MASS THREE-SPRING SYSTEM

We now proceed to analyze the system of equations (12.1.1) in two different cases. In Subsection 12.2.1, we show how to obtain the analytical solution for the two-mass three-spring system by using the standard matrix techniques of Linear Algebra. In Subsection 12.2.2, we show how to solve the same problem by turning it into an eigenvalue/eigenvector type of problem, which can be easily analyzed using the commands available in Mathematica and Python.

12.2.1 Equal Masses and Identical Springs - Analytical Solution

In the case of equal masses $m_1 = m_2 = m$, and identical spring constants $k_1 = k_2 = k_3 = k$, the equations of motion (12.1.1) become:

$$m\ddot{x}_1 = -kx_1 - k(x_1 - x_2) \qquad m\ddot{x}_2 = -k(x_2 - x_1) - kx_2 \tag{12.2.1}$$

This system of equations can be written in a compact matrix form:

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -2k & k \\ k & -2k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{12.2.2}$$

We can now solve this matrix equation by using the standard methods of Linear Algebra, and by following the same eigenvalue problem procedure we used in Chapter 11 for the principal moments of inertia.

We proceed in two steps, first we find the natural frequencies ω of the system, and second we find the positions $x_1(t)$ and $x_2(t)$, as follows.

Since we expect oscillatory motion, we try solutions of the form:

$$x_1(t) = A_1 e^{i\omega t} \qquad x_2(t) = A_2 e^{i\omega t} \tag{12.2.3}$$

where A_1 and A_2 are the unknown amplitudes of oscillation for the two masses, and ω is the unknown frequency of oscillation. Substituting these into the matrix equation (12.2.2):

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} -\omega^2 A_1 e^{i\omega t} \\ -\omega^2 A_2 e^{i\omega t} \end{pmatrix} = \begin{pmatrix} -2k & k \\ k & -2k \end{pmatrix}\begin{pmatrix} A_1 e^{i\omega t} \\ A_2 e^{i\omega t} \end{pmatrix} \tag{12.2.4}$$

By canceling the exponential factor $e^{i\omega t}$ which is common to all terms, and combining the matrices, we obtain:

$$\begin{pmatrix} -\omega^2 m + 2k & -k \\ -k & -\omega^2 m + 2k \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{12.2.5}$$

This matrix equation represents a system of equations. A theorem from Linear Algebra says that if the determinant of the matrix is nonzero, then there is a unique solution, which in this case is the trivial solution $A_1 = A_2 = 0$. However, in order for multiple solutions to exist, the determinant of the matrix must be zero. We are interested in a non-trivial solution $A_1, A_2 \neq 0$, so we solve for the values of ω which cause the determinant to be zero.

We set the determinant of the matrix equal to zero:

$$\det\begin{pmatrix} -\omega^2 m + 2k & -k \\ -k & -\omega^2 m + 2k \end{pmatrix} = 0 \tag{12.2.6}$$

$$\left(\omega^2 m - 2k\right)\left(\omega^2 m - 2k\right) - k^2 = 0 \tag{12.2.7}$$

Solving for ω, we obtain four possible solutions, only two of which are positive:

$$\omega_1 = \sqrt{\frac{k}{m}}, \quad \omega_2 = \sqrt{\frac{3k}{m}} \tag{12.2.8}$$

This completes the first part of the analysis, where we determined the two natural frequencies ω_1 and ω_2. In the previous section, we found that oscillatory solutions to our system of equations can have one of these two frequencies.

In order to complete the description of the system, we must also find the two unknown amplitudes of oscillation A_1 and A_2. If we substitute $\omega_1 = \sqrt{\frac{k}{m}}$ into the matrix equation (12.2.6), we obtain:

$$\begin{bmatrix} \left(\sqrt{\frac{k}{m}}\right)^2 m - 2k & k \\ k & \left(\sqrt{\frac{k}{m}}\right)^2 m - 2k \end{bmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{bmatrix} -k & k \\ k & -k \end{bmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0 \tag{12.2.9}$$

By multiplying the matrices, we obtain these two equations:

$$-kA_1 + kA_2 = 0 \tag{12.2.10}$$

$$kA_1 - kA_2 = 0 \tag{12.2.11}$$

It is clear that these two equations are identical, and that $A_1 = A_2$. Note that this always happens when we are finding the eigenvectors of a 2×2 matrix, one of the equations will be redundant and can just be ignored. We conclude that when $\omega_1 = \sqrt{k/m}$, the two amplitudes are equal $A_1 = A_2$, and therefore $x_1(t) = x_2(t) = A_1 e^{i\omega t}$. We can now write the first solution for the motion of the two masses, which corresponds to the first normal model $\omega_1 = \sqrt{k/m}$:

$$x_1(t) = x_2(t) = A_1 e^{i\omega_1 t} \tag{12.2.12}$$

Since $A_1 = A_2$ and $x_1(t) = x_2(t)$, this type of motion corresponds to both masses moving in the same direction and in phase at all times, as we saw previously in Figure 12.2a. This type of motion is known as the first *normal mode* or the *symmetric mode of oscillation*, and the general motion of the system in this mode can be written in terms of trigonometric functions:

$$x_1(t) = x_2(t) = D_1 \cos(\omega_1 t - \phi_1) \tag{12.2.13}$$

In matrix notation, the first normal mode can be written as:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = D_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_1 t - \phi_1) \tag{12.2.14}$$

By working in a similar fashion for the second natural frequency of the system, we substitute $\omega_2 = \sqrt{3k/m}$ into (12.1.1), and obtain $A_1 = -A_2$. Since $A_1 = -A_2$, this type of motion corresponds to the two masses moving in opposite directions, while the center of mass remains stationary, as shown in Figure 12.2b. This type of motion is known as the second *normal mode* or the *antisymmetric mode of oscillation*.

We can then write the second possible solution corresponding to $\omega_2 = \sqrt{3k/m}$ as:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = E_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_2 t - \phi_2) \tag{12.2.15}$$

In general, the motion of the system will be a linear combination of the two normal modes, corresponding to the frequencies $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{3k/m}$.

A faster method of obtaining the normal mode frequencies ω_1 and ω_2 and the amplitudes A_1 and A_2, is by solving the matrix equation (12.2.5) using the symbolic capabilities of Mathematica. Example 12.6 shows how to use the *Solve* command in Mathematica, to obtain the general solution for the two-mass three-spring system.

Example 12.6: Solving the two-mass three-spring system by solving the matrix equation

Solve the two oscillating mass system in Figure 12.1 as a matrix equation problem. Find the normal mode frequencies ω_1 and ω_2 and the general relationship between the amplitudes A_1 and A_2, in the case of equal masses and identical springs.

S***olution:***

The Mathematica code below uses the *Solve* and *Simplify* commands to solve the matrix equation (12.2.5). The *Simplify* command is used together with its option, *Assumptions* which restricts the results to positive values of the parameters k and m. After a warning message that Mathematica might not be able to obtain all solutions, the code produces four possible frequencies, only two of which are positive and therefore acceptable. The first normal mode frequency is $\omega_1 = \sqrt{k/m}$, and the corresponding relationship between the amplitudes is $A_1 = A_2$. This is of course the symmetric mode of oscillation for the system that we saw previously. The code also produces the second normal mode frequency $\omega_2 = \sqrt{3k/m}$, and the corresponding relationship between the amplitudes $A_1 = -A_2$ for the antisymmetric mode of oscillation.

$M = \begin{pmatrix} 2*k/m & -k/m \\ -k/m & 2*k/m \end{pmatrix}$;

Simplify$\left[\text{Solve}\left[M.\begin{pmatrix} \text{A1} \\ \text{A2} \end{pmatrix} == \omega\text{^}2*\begin{pmatrix} \text{A1} \\ \text{A2} \end{pmatrix}, \{\omega, \text{A1}, \text{A2}\}\right],\right.$
Assumptions->$k > 0$&&$m > 0$]

Solve : Equations may not give solutions for all "solve" variables.

OUTPUT:

$\left\{\{\text{A1} \to 0, \text{A2} \to 0\}, \left\{\omega \to -\sqrt{\frac{k}{m}}, \text{A2} \to \text{A1}\right\}, \left\{\omega \to \sqrt{\frac{k}{m}}, \text{A2} \to \text{A1}\right\},\right.$
$\left.\left\{\omega \to -\sqrt{3}\sqrt{\frac{k}{m}}, \text{A2} \to -\text{A1}\right\}, \left\{\omega \to \sqrt{3}\sqrt{\frac{k}{m}}, \text{A2} \to -\text{A1}\right\}\right\}$

12.2.2 Solving the Two-Mass and Three-Spring System as an Eigenvalue Problem

In this section, we show how to solve the two-mass three-spring system as an eigenvalue problem, using the commands available in Mathematica and Python. Let us consider again the general equations of motion (12.1.1), for the system of two masses in Figure 12.1:

$$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2) \qquad m_2\ddot{x}_2 = -k_2(x_2 - x_1) - k_3x_2 \tag{12.2.16}$$

or after dividing the first equation by m_1, and the second equation by m_2:

$$\ddot{x}_1 = -\frac{k_1}{m_1}x_1 - \frac{k_2}{m_1}(x_1 - x_2) \qquad \ddot{x}_2 = -\frac{k_2}{m_2}(x_2 - x_1) - \frac{k_3}{m_2}x_2 \tag{12.2.17}$$

By substituting a trial solution of the form $x_1(t) = A_1e^{i\omega t}$ and $x_2(t) = A_2e^{i\omega t}$ and canceling the common factor $e^{i\omega t}$, these equations yield:

$$-A_1\omega^2 = -\frac{k_1}{m_1}A_1 - \frac{k_2}{m_1}(A_1 - A_2) \qquad -A_2\omega^2 = -\frac{k_2}{m_2}(A_2 - A_1) - \frac{k_3}{m_2}A_2 \tag{12.2.18}$$

These can be written in compact matrix form as:

$$\begin{pmatrix} \frac{k_1+k_2}{m_1} & \frac{-k_2}{m_1} \\ \frac{-k_2}{m_2} & \frac{k_2+k_3}{m_2} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \omega^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \tag{12.2.19}$$

You will recognize that this equation is an *eigenvalue problem* in the theory of Linear Algebra, similar to the eigenvalue problems we encountered for the moment of inertia matrix in Chapter 11. As we remember from that chapter, in an eigenvalue problem we are given a *square matrix* **B***,* and we are asked to find a vector **a** such that $\mathbf{Ba} = \lambda\mathbf{a}$, where λ is a constant. The vector **a** is called an *eigenvector of the square matrix* **B***, corresponding to the eigenvalue* λ*.* We will explore the details of eigenvalue problems in Chapter 13.

For the two mass system of Figure 12.1, the eigenvalue problem to be solved becomes clear by writing (12.2.19) in this matrix form:

The Two-Mass System as an Eigenvalue Problem

$$\mathbf{G}\mathbf{A} = \omega^2 \mathbf{A} \tag{12.2.20}$$

$$\mathbf{A} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \qquad \mathbf{G} = \begin{pmatrix} \frac{k_1+k_2}{m_1} & \frac{-k_2}{m_1} \\ \frac{-k_2}{m_2} & \frac{k_2+k_3}{m_2} \end{pmatrix}$$

We are looking for the eigenvalues $\lambda = \omega^2$ of the square matrix $\mathbf{G}$, which will give us the natural frequencies of oscillation. We are also looking for the corresponding eigenvectors $\mathbf{A}$, which will give us the normal modes of oscillation corresponding to each natural frequency. In order for the eigenvalue equation (12.2.20) to have a non-trivial solution ($\mathbf{A} \neq \mathbf{0}$), the determinant of the matrix $\left(\boldsymbol{G} - \omega^2 \mathbf{1}\right)$ must be zero, where $\mathbf{1}$ is the 2×2 identity matrix. This gives:

$$\det \begin{bmatrix} \frac{k_1+k_2}{m_1} - \omega^2 & \frac{-k_2}{m_1} \\ \frac{-k_2}{m_2} & \frac{k_2+k_3}{m_2} - \omega^2 \end{bmatrix} = 0 \tag{12.2.21}$$

This is the characteristic equation of our eigenvalue problem. From this point on, we proceed by following the same two-step method used in the previous section. First, we must solve the characteristic equation (12.2.21) in order to find the frequencies ω_1 and ω_2. These frequencies will depend on k_1, k_2, k_3, m_1, and m_2. In the next step, we substitute the first natural frequency ω_1 into (12.2.19), in order to find (A_1, A_2), the first normal mode. Finally, we repeat the previous steps using the second natural frequency ω_2, in order to find (A_1, A_2) for the second normal mode.

The Mathematica and Python codes in Example 12.7 show how to find the eigenvectors and eigenvalues for the two-mass system in Figure 12.1.

Example 12.7: The two-mass system as an eigenvalue/eigenvector problem

Solve the two oscillating mass system in Figure 12.1 as an eigenvalue/eigenvector problem, in these two cases:

(a) Identical springs and identical masses.

(b) Identical springs and different masses.

Solution:

(a) In this case the matrix,

$$\mathbf{G} = \begin{pmatrix} \frac{2k}{m} & \frac{-k}{m} \\ \frac{-k}{m} & \frac{2k}{m} \end{pmatrix}$$

and the Mathematica code below uses the commands *Eigenvalues* and *Eigenvectors*. The code produces the first eigenvalue $\omega_1^2 = k/m$ or $\omega_1 = \sqrt{k/m}$, and the corresponding eigenvector,

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

This is exactly what we obtained in Section 12.2 for the first normal mode of the oscillating two-mass system. Similarly, the second eigenvalue $\omega_2 = \sqrt{3k/m}$, and the corresponding eigenvector

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

This is again the same result we obtained in Section 12.2 for the second normal mode.

$A = \begin{pmatrix} \frac{2*k}{m} & \frac{-k}{m} \\ -\frac{k}{m} & \frac{2*k}{m} \end{pmatrix}$;

Eigenvalues[A]//Simplify

OUTPUT: $\{\frac{3k}{m}, \frac{k}{m}\}$

Eigenvectors[A]//Simplify

OUTPUT: $\{\{-1,1\},\{1,1\}\}$

Here is the corresponding Python code, which produces the same eigenvalues and eigenvectors, each with a multiplicity of 1. For example the output $(k/m, 1, [Matrix([[1],[1]])]$ is interpreted as the $\omega_1 = \sqrt{k/m}$ with a multiplicity of 1, and the *Matrix* result indicates the corresponding eigenvector

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

```
from sympy import *
init_printing()
k = Symbol("k",positive="True")
m = Symbol("m",positive="True")

G=Matrix([[2*k/m,-k/m],[-k/m,2*k/m]])
a=G.eigenvals()
print("Eigenvalues=",list(a.keys()))
print("Eigenvectors=",G.eigenvects())

OUTPUT
Eigenvalues= [3*k/m, k/m]
Eigenvalues= [(k/m, 1, [Matrix([[1],[1]])]),
              (3*k/m, 1, [Matrix([[-1],[ 1]])])]
```

(b) In this case the matrix ,

$$\mathbf{G} = \begin{pmatrix} \frac{2k}{m_1} & \frac{-k}{m_1} \\ \frac{-k}{m_2} & \frac{2k}{m_2} \end{pmatrix}$$

and the Mathematica code produces the two eigenvalues and eigenvectors:

$$\omega_1 = \sqrt{\frac{k(m_1 + m_2 - z)}{m_1 m_2}} \qquad \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} \frac{m_1 - m_2 + z}{m_1} \\ 1 \end{pmatrix}$$

$$\omega_2 = \sqrt{\frac{k(m_1+m_2+z)}{m_1 m_2}} \qquad \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} \frac{-m_1+m_2+z}{m_1} \\ 1 \end{pmatrix}$$

where $z = \sqrt{m_1^2 - m_1 m_2 + m_2^2}$. In this two-mass three-spring system with unequal masses, the ratio of the amplitudes A_1/A_2 depends in a complicated manner on the masses (m_1, m_2), but does not depend on the spring constant k.

$A = \begin{pmatrix} \frac{2*k}{\text{m1}} & \frac{-k}{\text{m1}} \\ -\frac{k}{\text{m2}} & \frac{2*k}{\text{m2}} \end{pmatrix};$

Eigenvalues[A]//Simplify

OUTPUT: $\left\{\frac{k\left(\text{m1}+\text{m2}-\sqrt{\text{m1}^2-\text{m1m2}+\text{m2}^2}\right)}{\text{m1m2}}, \frac{k\left(\text{m1}+\text{m2}+\sqrt{\text{m1}^2-\text{m1m2}+\text{m2}^2}\right)}{\text{m1m2}}\right\}$

Eigenvectors[A]//Simplify

OUTPUT: $\left\{\left\{\frac{\text{m1}-\text{m2}+\sqrt{\text{m1}^2-\text{m1m2}+\text{m2}^2}}{\text{m1}},1\right\},\left\{-\frac{-\text{m1}+\text{m2}+\sqrt{\text{m1}^2-\text{m1m2}+\text{m2}^2}}{\text{m1}},1\right\}\right\}$

12.3 THE DOUBLE PENDULUM

Figure 12.3 shows a double pendulum, consisting of two masses m_1 and m_2 attached to massless rigid rods of lengths L_1 and L_2. We can treat this problem using the same methods as for the two-mass oscillating system, by developing the equations of motion and evaluating the natural frequencies. In Subsection 12.3.1, we will develop the Lagrangian and the equations of motion. In the Subsection 12.3.2, we will find the analytical solution for the special case of two identical coupled pendula. In Subsection 12.3.3, we will treat the double pendulum as an eigenvalue problem, and show how to obtain the natural frequencies and the amplitudes of the normal modes.

12.3.1 The Lagrangian and Equations of Motion - Numerical Solutions

The position (x_1, y_1) of the first mass is $(x_1, y_1) = (L_1 \sin\theta_1, L_1 \cos\theta_1)$, so that the kinetic energy of mass m_1 is:

$$T_1 = \frac{1}{2} m_1 \left(\dot{x}_1^2 + \dot{y}_1^2\right) = \frac{1}{2} m_1 \left(L_1^2 \cos^2\theta_1 \dot{\theta}_1^2 + L_1^2 \sin^2\theta_1 \dot{\theta}_1^2\right) = \frac{1}{2} m_1 L_1^2 \dot{\theta}_1^2 \tag{12.3.1}$$

The location (x_2, y_2) of the second mass is shifted with respect to the first mass by (x_1, y_1), so that $(x_2, y_2) = (L_1 \sin\theta_1 + L_2 \sin\theta_2, L_1 \cos\theta_1 + L_2 \cos\theta_2)$. Therefore the kinetic energy of the second mass is:

$$T_2 = \frac{1}{2} m_2 \left(\dot{x}_2^2 + \dot{y}_2^2\right) \tag{12.3.2}$$

$$= \frac{1}{2} m_2 \left\{\frac{d}{dt}\left(L_1 \sin\theta_1 + L_2 \sin\theta_2\right)^2 + \frac{d}{dt}\left(L_1 \cos\theta_1 + L_2 \cos\theta_2\right)^2\right\} \tag{12.3.3}$$

After differentiating and collecting terms, we find:

$$T_2 = \frac{1}{2}m_2\left(L_1^2\dot{\theta}_1^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) + L_2^2\dot{\theta}_2^2\right) \tag{12.3.4}$$

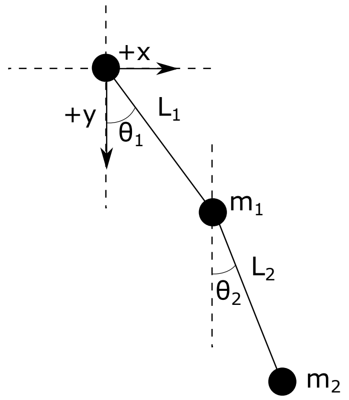


Figure 12.3: The double pendulum oscillator is characterized by the two angles $(\theta_1(t), \theta_2(t))$.

The total kinetic energy is then:

$$T = \frac{1}{2}(m_1+m_2)L_1^2\dot{\theta}_1^2 + m_2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) + \frac{1}{2}m_2L_2^2\dot{\theta}_2^2 \tag{12.3.5}$$

For amplitudes of small oscillations we use the approximation $\cos(\theta_1-\theta_2) \simeq 1$, so that:

$$T = \frac{1}{2}(m_1+m_2)L_1^2\dot{\theta}_1^2 + m_2L_1L_2\dot{\theta}_1\dot{\theta}_2 + \frac{1}{2}m_2L_2^2\dot{\theta}_2^2 \tag{12.3.6}$$

The total potential energy V is the sum of potential energies for each pendulum:

$$V = m_1gy_1 + m_2gy_2 = (m_1+m_2)gL_1\cos\theta_1 + m_2gL_2\cos\theta_2 \tag{12.3.7}$$

For amplitudes of small oscillations, we use the approximation $\cos\theta_1 \simeq 1-\frac{\theta_1^2}{2}$ and $\cos\theta_2 \simeq 1-\frac{\theta_2^2}{2}$ so that:

$$V = (m_1+m_2)gL_1\left(1-\frac{\theta_1^2}{2}\right) + m_2gL_2\left(1-\frac{\theta_2^2}{2}\right) \tag{12.3.8}$$

The Lagrangian $L = T - V$ of the system is:

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}(m_1+m_2)L_1^2\dot{\theta}_1^2 + m_2L_1L_2\dot{\theta}_1\dot{\theta}_2 + \frac{1}{2}m_2L_2^2\dot{\theta}_2^2 \\ &-(m_1+m_2)gL_1\left(1-\frac{\theta_1^2}{2}\right) - m_2gL_2\left(1-\frac{\theta_2^2}{2}\right)\end{aligned} \tag{12.3.9}$$

We can now find the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1}\right) - \frac{\partial\mathcal{L}}{\partial\theta_1} = 0 \qquad \left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2}\right) - \frac{\partial\mathcal{L}}{\partial\theta_2} = 0 \tag{12.3.10}$$

By evaluating the derivatives and simplifying, we obtain: